# $N$-point functions of the KdV hierarchy and higher Weil–Petersson volumes

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# 1. Introduction

i) What is an intersection number?
ii) What is an integrable system?
iii) Witten's conjecture.
iv) Our results.

## Part i) What is an intersection number?

Around the year 1850, in his study of multi-valued functions, Riemann introduced the notion of Riemann surfaces.

Riemann proves that the dimension of

$$\mathcal{M}_g = \text{moduli space of Riemann surfaces of genus } g$$

is $3g - 3$, $g \geq 2$.

Denote by $\mathcal{M}_{g,N}$ the moduli space of genus $g$ smooth algebraic curves with $N$ marked points.

According to Riemann, we have

$$\dim_{\mathbb{C}} \mathcal{M}_{g,N} = 3g - 3 + N, \qquad g \geq 2.$$

In 1969, Deligne and Mumford considered a compactification of $\mathcal{M}_{g,N}$.

They added in $\mathcal{M}_{g,N}$ certain boundary points, which are singular algebraic curves allowing nodal singular points. This is known as the Deligne–Mumford compactification, denoted by $\overline{\mathcal{M}}_{g,N}$.

Let $\mathcal{L}_i$ be the $i_{\text{th}}$ tautological line bundle over $\overline{\mathcal{M}}_{g,N}$. The fiber of $\mathcal{L}_i$ at a point $[(\Sigma_g, p_1, \ldots, p_N)] \in \overline{\mathcal{M}}_{g,N}$ is defined as $T^*_{p_i} \Sigma_g$, $i = 1, \ldots, N$.

Let

$$\psi_i := c_1(\mathcal{L}_i), \qquad i = 1, ..., N$$

be the first Chern classes of $\mathcal{L}_i$, called the $\psi-$classes.

The intersection numbers of $\psi-$classes on $\overline{\mathcal{M}}_{g,N}$ are certain non-negative rational numbers defined by the integrals

$$\langle \tau_{k_1} \cdots \tau_{k_N} \rangle_{g,N} := \int_{\overline{\mathcal{M}}_{g,N}} \psi_1^{k_1} \wedge \cdots \wedge \psi_N^{k_N}, \qquad k_1, \ldots, k_N \geq 0.$$

For examples, $\int_{\overline{\mathcal{M}}_{0,3}} 1 = 1$; $\int_{\overline{\mathcal{M}}_{51,2}} \psi_1^{61} \wedge \psi_2^{91} = A_1/A_2$ with

$$\begin{aligned} A_1 &= 938605017283641258750098935902440374327740322001634337 9, \\ A_2 &= 1295911182815633150530109902582474075123568533734585207009074641418782551472383 3 \\ &\quad 9379331556666041363562054992071393762192115131342143042355200000000000; \end{aligned}$$

$$\langle \tau_{20}\tau_{21}\tau_{22} \rangle = \frac{5990793025211453654394615727 1}{34410236643719662106010647646034081605299994624000 0}.$$

From the dimension counting, we know that $\langle \tau_{k_1} \cdots \tau_{k_N} \rangle_{g,N}$ is zero unless

$$k_1 + \cdots + k_N = 3g - 3 + N.$$

We collect the intersection numbers into certain generating functions

$$\mathcal{F}_g(\mathbf{t}) := \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{k_1,\dots,k_N \geq 0} \langle \tau_{k_1} \cdots \tau_{k_N} \rangle_{g,N} \, t_{k_1} \cdots t_{k_N} \, ,$$

called the genus $g$ free energies, $g \geq 0$. Here, $\mathbf{t} = (t_0, t_1, t_2, \dots)$.

Let us further collect $\mathcal{F}_g$, $g \geq 0$ into

$$Z(\mathbf{t}; \epsilon) = \exp \sum_{g=0}^{\infty} \epsilon^{2g-2} \mathcal{F}_g(\mathbf{t}).$$

Then $Z$ is called the partition function of intersection numbers.

The nature of intersections numbers sheds light on several interesting areas of mathematics including for example

(1) Lie algebra;

(2) Complex analysis;

(3) Random matrices;

(4) Quantum singularity theory;

(5) Integrable systems;

etc. Our main concern here is on the aspect of integrable systems.

## Part ii) What is an integrable system?

Integrability is an important notion in differential geometry.

One of the origins of integrability arises from the problem of *testing whether a collection of vectors fields could be served as a coordinate system.*

Answers to this problem or to its analogous problems date back to Clebsch, Frobenius, etc., during the period 1840–1877.

Clebsch–Frobenius' Theorem. *Given a smooth manifold* $M^n$ *and given any* $k$*-dimensional distribution* $D^k = [X_1, \dots, X_k]$, $X_i \in C^\infty(M, TM)$, $i = 1, \dots, k$. *There exists a local coordinate system* $w^1, \dots, w^k$ *such that*

$$D^k = [\partial_{w^1}, \dots, \partial_{w^k}]$$

*i.f.f.*

$$[X_i, X_j] \in D^k, \quad \forall\, i, j = 1, \dots, k.$$

Depending on context, Clebsch–Frobenius' theorem could have various versions.

For example, in the theory of finite-dimensional symplectic geometry, we have Arnold–Liouville's theorem.

Our interest is on integrability for infinite-dimensional manifolds, i.e. on the integrability for PDEs.

Let us first recall some history about the discovery of this interest.

In 1834, around the similar time with Riemann's pioneer work, John Scott Russell wrote in his diary:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped — not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation." (c.f. Wikipedia)

This phenomenon is nowadays called a *soliton*.

Mathematical interpretation of such a phenomenon becomes an interesting question for applied mathematicians in the 19th century.

Boussinesq (in 1877), and independently Korteweg and de Vries (in 1895) derived a nonlinear PDE for describing the motion of (1+1)-dimensional water waves

$$u_t = uu_x + \frac{\epsilon^2}{12} u_{xxx},$$

now known as the KdV equation. Here, $\epsilon$ is a complex parameter, $t$ is called the time variable, and $x$ the space variable.

Russell's observation could be explained by the fact that the KdV equation admits a one-soliton solution:

$$u(x,t) = b_1^2\,\epsilon^2\,\mathrm{Sech}^2\left[b_1\left(x + \frac{1}{3}b_1\epsilon^2 t - b_0\right)\right],\ \forall\, b_0, b_1 \in \mathbb{C}.$$

The number $\epsilon$ appearing in KdV is called a *dispersion* parameter.

Consider the simplest situation such that $\epsilon = 0$.

In this case, the KdV equation becomes

$$u_t = u\, u_x.$$

This simpler equation is called the Riemann–Hopf equation.

Let us study the initial value problem of the Riemann–Hopf equation

$$u_t = u\,u_x,$$
$$u|_{t=0} = f(x).$$

Closed form of solution to this problem has been well-known since Riemann

$$x + t\,u = f^{-1}(u).$$

Assuming $f(x)$ is a Schwarzian type function over $\mathbb{R}$, we find that as $t$ increases, $u(x,t)$ becomes multi-valued. That is, the wave breaks.

For a nonlinear evolutionary PDE, wave breaking is a common phenomenon, which is referred to as a nonlinear effect.

In other words, the fact that KdV possesses soliton solutions is an uncommon phenomenon.

So, for the one-soliton solution of KdV, $\epsilon$ plays the role such that the nonlinear effect gets balanced with the dispersion effect.

For a general solution of KdV, single-valuedness can also be kept for large $t$, but oscillation would occur from a certain instant.

Let us now go back to the introduction of integrability.

An evolutionary PDE

$$u_s = g(u; u_x, u_{xx}, \dots)$$

is called a symmetry of the Riemann equation if

$$u_{st} = u_{ts}.$$

## Example 1.1

$$u_{t_n} := \frac{u^n}{n!} u_x, \quad n = 0, 1, \dots$$

are symmetries of the Riemann–Hopf equation.

The interesting fact is that the PDEs

$$u_{t_n} = \frac{u^n}{n!} u_x, \quad n = 0, 1, \dots$$

are mutually symmetries, i.e.

$$\left[\partial_{t_n}, \partial_{t_m}\right](u) = 0, \quad \forall\, n, m \geq 0.$$

So these PDEs can be solved together. They are called the Riemann hierarchy.

The Riemann–Hopf equation is also an infinite dimensional Hamiltonian system

$$u_t = P\frac{\delta H}{\delta u(x)},$$

where $P$ is a Hamiltonian operator defined by

$$P = \partial_x$$

and $H$ is a local Hamiltonian defined by

$$H = \frac{1}{6}\int_{\mathbb{R}} u(x)^3 \, dx$$

According to Nöther, the infinitely many symmetries we obtained above give rise to infinitely many conservation laws of the Riemann equation

$$H_k = \frac{1}{(k+2)!} \int_{\mathbb{R}} u(x)^{k+2} \, dx, \qquad k = 0, 1, \ldots.$$

Moreover, it is straightforward to check that $H_k$, $k \geq 0$ are conservation laws of each member of the Riemann hierarchy, i.e. we have

$$\frac{\partial H_k}{\partial t_m} = 0, \qquad m \geq 0.$$

This vaguely indicates that each member of the Riemann hierarchy is an integrable PDE.

The Riemann hierarchy gives the first example of an integrable hierarchy.

## Lemma 1.2

*Consider the following family of evolutionary PDEs*

$$u_{t_k} = \frac{u^k}{k!} u_x + \sum_{j=1}^{\infty} P_{k,j}(u; u_x, \ldots; \epsilon)\, \epsilon^j, \qquad P_{k,j} \in \mathcal{A}_u,\ k \geq 0,$$

$$\deg P_{k,j} = j,$$

$$u_{t_1} = u u_x + \frac{\epsilon^2}{12} u_{xxx}.$$

*There exist unique differential polynomials* $P_{k,j}$, $j \geq 0$ *such that*

$$[\partial_{t_k}, \partial_{t_1}](u) = 0, \quad k \geq 0.$$

The uniquely family in the above lemma is called the KdV hierarchy.

Again, members of the KdV hierarchy are mutually symmetries

$$[\partial_{t_k}, \partial_{t_m}]\,(u) = 0, \quad \forall\, k, m \geq 0.$$

This roughly indicates that the KdV equation is integrable, as well as that each member of the KdV hierarchy is integrable.

Moreover, due to analogue of the Clebsch–Frobenius' theorem, the infinitely many equations of the KdV hierarchy can be solved together.

The KdV hierarchy gives the second example of an integrable hierarchy.

Part iii) Witten's conjecture.

The study of the deep relation between intersection numbers and integrable hierarchies was initiated by Witten.

**Witten's conjecture**[1] Let $Z(\mathbf{t};\epsilon)$ be the partition function of intersection numbers and let $u = \epsilon^2 \partial_{t_0}^2 \log Z$, then

a) $u$ satisfies the *KdV hierarchy*. Particularly,

$$\frac{\partial u}{\partial t_1} = u \frac{\partial u}{\partial t_0} + \frac{1}{12}\epsilon^2 \frac{\partial^3 u}{\partial t_0^3}.$$

b) $Z$ satisfies the *string equation*

$$\frac{\partial Z}{\partial t_0} = \frac{t_0^2}{2\epsilon^2} Z + \sum_{k=0}^{\infty} t_{k+1} \frac{\partial Z}{\partial t_k}.$$

[1] E. Witten. In "Surveys in differential geometry," 1991: 243-310.

In 1992, Kontsevich[2] proved Witten's conjecture. The partition function $Z$ is now usually called the Witten–Kontsevich tau function.

[2] M. Kontsevich. Comm. Math. Phys., 1992, 147: 1-23.

## Part iv) Our results.

Theorem 1.3 (Bertola–Dubrovin–Y, 2015, arXiv: 1504.06452)

*The (descendent) Gromov–Witten invariants of a point in full genera can be generated by the exponential function*

$$\exp\left(\frac{x^3}{24}\right)$$

*in a purely algebraic way, expressed by simple generating formulae.*

## Theorem 1.4 (Bertola–Dubrovin–Y, 2015, arXiv: 1508.03750)

*Witten's (descendent) $3$-spin intersection numbers in full genera can be determined in a purely algebraic way by two confluent hypergeometric limit functions*

$$\phi = x^{-\frac{2}{3}} \cdot {}_0F_1\left(; \frac{1}{3}; -\frac{x^8}{1728}\right), \quad \chi = x^{-\frac{1}{3}} \cdot {}_0F_1\left(; \frac{2}{3}; -\frac{x^8}{1728}\right).$$

Here we recall that

$${}_0F_1(; a; z) := \sum_{k=0}^{\infty} \frac{1}{(a)_k} \frac{z^k}{k!}$$

where $(a)_k$ is the Pochhammer symbol, i.e.

$$(a)_k := a(a+1)\cdots(a+k-1).$$

Note that $\phi$ and $\chi$ can be uniquely characterized by the following ODEs

$$27\,x^2\,\phi'' - 81\,x\,\phi' + \left(x^8 - 84\right)\,\phi = 0,$$

$$27\,x^2\,\chi'' - 27\,x\,\chi' + \left(x^8 - 21\right)\,\chi = 0,$$

respectively.

# 2. Wave function and $N$-point functions of KdV

Our goal is to use the Witten–Kontsevich theorem, i.e. the KdV hierarchy to the study of intersection numbers.

These numbers become difficult as the genus grows or as the number of marked points increases.

How to resolve the difficulty?

Recall that for a linear evolutionary PDE, the evolution in Fourier's spectral world is often easier than evolution in the "real" world.

It is known from 1970s that the KdV hierarchy admits a nonlinear Fourier transform, also called a Lax pair.

These two facts suggest us to go to the spectral world of KdV.

It will turn out that in the spectral world we have easy numbers which contain enough information for the reconstruction of all intersection numbers.

These "easy" numbers will be coefficients of the Faber–Zagier series:

$$
\begin{aligned}
c(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{288^k} \frac{(6k)!}{(3k)!(2k)!} z^{-3k}, \\
q(z) &= \sum_{k=0}^{\infty} \frac{1+6k}{1-6k} \frac{(-1)^k}{288^k} \frac{(6k)!}{(3k)!(2k)!} z^{-3k}.
\end{aligned}
$$

Later we will also see that $z$ is the spectral parameter, and $c$, $q$ are the wave functions of KdV for the Witten–Kontsevich solution.

So the final difficulty reduces to the reconstruction of the tau-function from a wave function.

There have been two main techniques: the inverse scattering method and the Sato–Segal–Wilson Grassmannian method.

The inverse scattering method is suitable for analytic solutions, but not suitable for formal series solution.

The Sato Grassmannian approach reconstructs the Witten–Kontsevich tau-function in the following explicit form

$$Z = \sum_{\mu \in \mathbb{Y}} \pi_\mu \, \mathrm{sch}_\mu(T), \quad T = (T_1, T_2, \dots)$$

called the Sato–Zhou formula ([M. Sato, Zhou, Balogh-Y]). Here, $T_{2k+1} = t_k$, and $\pi_\mu$ is the Plücker coordinate associated to $\mu$, which can be uniquely determined by $c(z), q(z)$.

However, we want to find a direct reconstruction of $\log Z$, instead of $Z$.

To this end, we introduce a new approach for the reconstruction.

Recall that the $k$-th member of the KdV hierarchy can be described by

$$L_{t_k} = [A_k, L],$$
$$A_k := \frac{1}{(2k+1)!!} \left(L^{\frac{2k+1}{2}}\right)_+, \quad k \geq 0.$$

Here

$$L = \partial_x^2 + 2\,u(x)$$

is called the Lax operator; and we have taken $\epsilon = 1$ for simplicity.

Namely, the KdV hierarchy are compatibility conditions for the following linear PDEs

$$\begin{aligned} L(\psi) &= z^2 \psi, \\ \psi_{t_k} &= A_k \psi. \end{aligned}$$

$(L, A_k)$ are called Lax pairs.

Solutions $\psi = \psi(z; t_0, t_1, \ldots, )$ of these linear PDEs are called wave functions of the KdV hierarchy.

Let $\mathcal{B} = \mathbb{C}[[t_k,\, k = 0, 1, 2, \ldots]]$ be the Bosonic Fock space.

A tau-function $\tau(\mathbf{t})$ of the KdV hierarchy is an element in $\mathcal{B}$ satisfying the Hirota bilinear identities: $\forall\, \mathbf{t},\, \widetilde{\mathbf{t}},\, p \geq 0$,

$$\operatorname*{res}_{z=\infty} z^{2p}\, \tau(\mathbf{t} - [z^{-1}])\, \tau(\widetilde{\mathbf{t}} + [z^{-1}]) \exp\left( \sum_{j \geq 1} \frac{t_j - \widetilde{t}_j}{(2j+1)!!} z^{2j+1} \right) dz = 0.$$

Here, $\mathbf{t} - [z^{-1}] := \left( t_0 - z^{-1}, \ldots, t_k - \frac{(2k-1)!!}{z^{2k+1}}, \ldots \right).$

Let $\tau(\mathbf{t})$ be any tau-function. It is proved by Sato that

$$\psi(z;\mathbf{t}) := \frac{\tau(\mathbf{t}-[z^{-1}])}{\tau(\mathbf{t})} \mathrm{e}^{\vartheta(z;\mathbf{t})}, \quad \vartheta(z;\mathbf{t}) := \sum_{j=0}^{\infty} t_j \frac{z^{2j+1}}{(2j+1)!!}.$$

is a particular wave function of the KdV hierarchy. The function

$$\psi^*(z;\mathbf{t}) := \psi(-z;\mathbf{t}) = \frac{\tau(\mathbf{t}+[z^{-1}])}{\tau(\mathbf{t})} \mathrm{e}^{-\vartheta(z;\mathbf{t})}$$

is called a dual wave function.

We have the following question: Given $\psi(z;\mathbf{t})$, can we reconstruct $\log\tau(\mathbf{t})$?

## Definition 2.1

For any tau-function $\tau(\mathbf{t})$ of the KdV hierarchy, we call the functions

$$\langle\langle \tau_{k_1} \tau_{k_2} \dots \tau_{k_N} \rangle\rangle(\mathbf{t}) := \frac{\partial^N \log \tau}{\partial t_{k_1} \dots \partial t_{k_N}}(\mathbf{t}), \quad N \geq 1$$

and the numbers

$$\langle \tau_{k_1} \tau_{k_2} \dots \tau_{k_N} \rangle := \langle\langle \tau_{k_1} \tau_{k_2} \dots \tau_{k_N} \rangle\rangle(\mathbf{t} = 0)$$

the $N$-point correlation functions and the $N$-point correlators of the tau-function, respectively.

We now collect the correlation functions into generating function. For any $N \geq 1$, we define the $N$-point function of correlation functions by

$$F(z_1, \ldots, z_N; \mathbf{t}) := \sum_{k_1, \ldots, k_N = 0}^{\infty} \langle\langle \tau_{k_1} \ldots \tau_{k_N} \rangle\rangle(\mathbf{t}) \, \frac{(2k_1+1)!!}{z_1^{2k_1+2}} \cdots \frac{(2k_N+1)!!}{z_N^{2k_N+2}}.$$

Evaluating it at $\mathbf{t} = 0$

$$F(z_1, \ldots, z_N) := F(z_1, \ldots, z_N; \mathbf{0})$$

one obtains the $N$-point functions of correlators.

## Theorem 2.2 (Bertola–Dubrovin–Y, 2015, arXiv: 1504.06452)

*Let $\tau$ be any tau-function of the KdV hierarchy and let $\psi, \psi^*$ be the corresponding (uniquely defined) wave and dual wave functions. We have*

$$F(z;\mathbf{t}) = \frac{1}{2}\mathrm{Tr}\left(\Psi^{-1}(z)\Psi_z(z)\sigma_3\right) - \vartheta_z(z)$$

*where $\Psi$ is the matrix-valued wave function defined by*

$$\Psi(z;\mathbf{t}) = \begin{pmatrix} \psi(z;\mathbf{t}) & \psi^*(z;\mathbf{t}) \\ -\psi_x(z;\mathbf{t}) & -\psi^*_x(z;\mathbf{t}) \end{pmatrix}$$

*and* $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *is the Pauli matrix.*

Proof is not difficult by using the Hirota bilinear equations.

The problem is solved.

The new question is the following: Given $\psi(z; \mathbf{t} = \mathbf{0})$, $\psi_x(z; \mathbf{t} = \mathbf{0})$, can we reconstruct $\log \tau(\mathbf{t})$?

## Theorem 2.3 (Bertola–Dubrovin–Y, 2015, arXiv: 1504.06452)

*Let* $\tau$ *be any tau-function of the KdV hierarchy. Define*

$$\Theta(z;\mathbf{t}) = z\,\Psi(z;\mathbf{t})\,\sigma_3\,\Psi^{-1}(z;\mathbf{t})$$

*Then the generating function of* $N$*-point correlation functions with* $N \geq 2$ *has the following expression*

$$F(z_1,\dots,z_N;\mathbf{t}) = -\frac{1}{N}\sum_{r\in S_N}\frac{\mathrm{Tr}\left(\Theta(z_{r_1})\cdots\Theta(z_{r_N})\right)}{\prod_{j=1}^{N}(z_{r_j}^2 - z_{r_{j+1}}^2)} - \delta_{N,2}\frac{z_1^2+z_2^2}{(z_1^2-z_2^2)^2}.$$

*Here* $S_N$ *denotes the group of permutations of* $\{1,\dots,N\}$.

The new question is answered by taking $\mathbf{t}=\mathbf{0}$.

Let $\mathcal{R} = \psi(z;\mathbf{t}) \cdot \psi^*(z;\mathbf{t})$. We have

## Lemma 2.4

*The function $\mathcal{R}$ satisfies*

$$\mathcal{R}\,\mathcal{R}_{xx} - \frac{1}{2}\,\mathcal{R}_x^2 + (4\,u - 2\,z^2)\,\mathcal{R}^2 = -2\,z^2,$$
$$\mathcal{R} = 1 + \mathcal{O}(z^{-1}), \quad z \to \infty.$$

*Moreover, the solution to the above ODE boundary value problem in the ring $\mathcal{A}_u[[z^{-1}]]$ is unique.*

The traceless matrix $\Theta$ in the main theorem can be written as

$$\Theta(z;\mathbf{t}) = \frac{1}{2}\left[\begin{array}{cc} -\mathcal{R}_x & -2\mathcal{R} \\ \mathcal{R}_{xx} - 2(z^2 - 2\,u)\mathcal{R} & \mathcal{R}_x \end{array}\right].$$

In particular, from the main theorem we know that all multi-point correlation functions belong to $\mathcal{A}_u$.

We have obtained a simple way of reconstruction of $\log \tau(\mathbf{t})$ from $\psi(z; \mathbf{t} = \mathbf{0})$, $\psi_x(z; \mathbf{t} = \mathbf{0})$.

However, how to solve the initial wave function?

Idea is simply by putting the constraint of the string equation from tau-function to wave functions.

Recall that a general string equation reads

$$L_{-1}\tau = 0, \qquad L_{-1} := \sum_{k\geq 0} \tilde{t}_{k+1}\frac{\partial}{\partial t_k} + \frac{1}{2}\tilde{t}_0^2$$

where $\tilde{t}_k = t_k - c_k,\ k \geq 0$, with arbitrary constants $c_k$. E.g., for the Witten–Kontsevich solution $c_1 = 1$, $c_k = 0$ for $k \neq 1$.

### Definition 2.5

We define the generalized Kac–Schwarz operator by

$$S_z = \frac{1}{z}\partial_z - \frac{1}{2z^2} - \sum_{k=0}^{\infty} \frac{c_k}{(2k-1)!!} z^{2k-1}.$$

We have the following generalized Buryak's lemma:

### Lemma 2.6

*Let* $K(z;\mathbf{t}) := \psi(z;\mathbf{t}) \cdot \tau(\mathbf{t}) = \tau(\mathbf{t} - [z^{-1}])\,\mathrm{e}^{\vartheta(z)}$. *The following formula holds true:*

$$L_{-1}K = S_z K.$$

We proved that this equation implies strong and simple ODE which uniquely specify $\psi(z; t_0, t_1 = 0, t_2 = 0, \dots)$ and $\psi_x(z; t_0, 0, \dots)$.

# 3. Application to intersection numbers of $\psi$-classes

To apply the main theorem to intersection numbers, we only need to derive a closed form of the initial datum of the wave function.

Recall that a simple corollary of the string equation is

$$u^{WK}(x, 0, 0, \dots) = x.$$

Let $f^{WK}(z; x) = \psi^{WK}(z; x, 0, \dots)$. Then we have

$$f^{WK}_{xx} + 2\,x f^{WK} = z^2 f^{WK}.$$

According to the definition of the wave function, $f^{WK}(z; x)$ satisfies the asymptotic behaviour

$$f^{WK}(z; x) = \big(1 + \mathcal{O}(z^{-1})\big)\exp(xz), \qquad z \to \infty.$$

Solving the above ODE boundary value problem we obtain

$$f^{WK}(z;x) = G(z)\sqrt{2\pi z}\,\mathrm{e}^{\frac{z^3}{3}}\,2^{\frac{1}{3}}\,\mathrm{Ai}\left(2^{-\frac{2}{3}}\,(z^2 - 2\,x)\right)$$

where $G(z)$ is an arbitrary function in $z$ of the form

$$G(z) = 1 + \mathcal{O}(z^{-1}).$$

To fix the function $G(z)$ we need to use the string equation

$$\sum_{k=0}^{\infty} t_{k+1} \frac{\partial Z}{\partial t_k} + \frac{t_0^2}{2} Z = \frac{\partial Z}{\partial t_0}.$$

Let $S_z$ be the corresponding Kac–Schwarz operator

$$S_z := \frac{1}{z}\partial_z - \frac{1}{2z^2} - z.$$

Then the string equation implies the following equation of Buryak

$$S_z^{WK} f^{WK}(z;x) = -\partial_x f^{WK}(z;x).$$

Substituting

$$f^{WK}(z;x) = G(z)\sqrt{2\pi z}\,\mathrm{e}^{\frac{z^3}{3}}\,2^{\frac{1}{3}}\,\mathrm{Ai}\left(2^{-\frac{2}{3}}\,(z^2 - 2\,x)\right)$$

in Buryak's equation and comparing both sides we find

$$G(z) \equiv 1.$$

Define

$$
\begin{aligned}
A^{WK}(z) &:= \psi^{WK}(z;\mathbf{0}) = f^{WK}(z;0), \\
B^{WK}(z) &:= \psi_x^{WK}(z;\mathbf{0}) = f_x^{WK}(z;0).
\end{aligned}
$$

The above derivations imply

$$
\begin{aligned}
A^{WK}(z) &= c(z), \\
B^{WK}(z) &= z\,q(z).
\end{aligned}
$$

Here $c(z)$ and $q(z)$ are the Faber–Zagier series

$$
\begin{aligned}
c(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{288^k} \frac{(6k)!}{(3k)!(2k)!}\, z^{-3k}, \\
q(z) &= \sum_{k=0}^{\infty} \frac{1+6k}{1-6k} \frac{(-1)^k}{288^k} \frac{(6k)!}{(3k)!(2k)!}\, z^{-3k}.
\end{aligned}
$$

Substituting $A^{WK}$, $B^{WK}$ in

$$\Theta(z;\mathbf{t}) = z\,\Psi(z;\mathbf{t})\,\sigma_3\,\Psi^{-1}(z;\mathbf{t})$$

of the main theorem, we obtain an explicit generating function of intersection numbers of $\psi-$classes.

## Theorem 3.1

*Define the $N$-point functions of intersection numbers of $\psi$-classes by*

$$F^{WK}(z_1,\dots,z_N) := \sum_{k_1,\dots,k_N\ge 0} \langle \tau_{k_1}\dots\tau_{k_N}\rangle \frac{(2k_1+1)!!}{z_1^{2k_1+2}}\cdots\frac{(2k_N+1)!!}{z_N^{2k_N+2}}.$$

*Let $M(z)$ denote the following matrix-valued formal series*

$$M(z) = \frac{1}{2}\begin{pmatrix} -\sum_{g=1}^{\infty} \frac{(6g-5)!!}{24^{g-1}\cdot(g-1)!} z^{-6g+4} & -2\sum_{g=0}^{\infty} \frac{(6g-1)!!}{24^{g}\cdot g!} z^{-6g} \\ 2\sum_{g=0}^{\infty} \frac{6g+1}{6g-1}\frac{(6g-1)!!}{24^{g}\cdot g!} z^{-6g+2} & \sum_{g=1}^{\infty} \frac{(6g-5)!!}{24^{g-1}\cdot(g-1)!} z^{-6g+4} \end{pmatrix}.$$

*Then we have*

$$F^{WK}(z) = \sum_{g=1}^{\infty} \frac{(6g-3)!!}{24^g\cdot g!} z^{-(6g-2)},$$

$$F^{WK}(z_1,\dots,z_N) = -\frac{1}{N}\sum_{r\in S_N} \frac{\mathrm{Tr}\left(M(z_{r_1})\cdots M(z_{r_N})\right)}{\prod_{j=1}^{N}(z_{r_j}^2 - z_{r_{j+1}}^2)} - \delta_{N,2}\frac{z_1^2+z_2^2}{(z_1^2-z_2^2)^2}.$$

Proof. By using the Witten–Kontsevich theorem, the main theorem, and the following combinatorial identities:

$$
\begin{aligned}
c(z)\cdot q(-z)+c(-z)\cdot q(z) &= 2,\\
c(z)\cdot c(-z) &= \sum_{g=0}^{\infty}\frac{(6g-1)!!}{24^g\cdot g!}z^{-6g},\\
q(z)\cdot q(-z) &= -\sum_{g=0}^{\infty}\frac{6g+1}{6g-1}\frac{(6g-1)!!}{24^g\cdot g!}z^{-6g},\\
c(z)\cdot q(-z) &= 1-\frac{1}{2}\sum_{g=1}^{\infty}\frac{(6g-5)!!}{24^{g-1}\cdot (g-1)!}z^{-6g+3},\\
q(z)\cdot c(-z) &= 1+\frac{1}{2}\sum_{g=1}^{\infty}\frac{(6g-5)!!}{24^{g-1}\cdot (g-1)!}z^{-6g+3}.
\end{aligned}
$$

# 4. Application to higher Weil–Petersson volumes

The *Main Theorem* also allows us to compute higher Weil–Petersson volumes, by which we mean integrals of mixed $\psi$- and $\kappa$-classes of the form

$$\langle \kappa_1^{d_1} \dots \kappa_\ell^{d_\ell} \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n} \kappa_1^{d_1} \dots \kappa_\ell^{d_\ell}.$$

Recall that $\kappa$-classes are defined by

$$\kappa_i = f_* \left( \psi_{n+1}^{i+1} \right) \in A^i \left( \overline{\mathcal{M}}_{g,n} \right), \quad i \geq 1$$

where $f : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal curve (the forgetful map).

By dimension counting, $\langle \kappa_1^{d_1} \dots \kappa_\ell^{d_\ell} \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n}$ take zero values unless

$$\sum_{j=1}^{n} k_j + \sum_{j=1}^{l} j\, d_j = 3g - 3 + n.$$

Similarly as before, people collect the higher Weil–Petersson volumes into the following generating function $Z^\kappa(\mathbf{t};\mathbf{s})$

$$Z^\kappa(\mathbf{t};\mathbf{s}) = \exp\left(\sum_{g,N,\ell\geq 0}\frac{1}{N!}\sum_{d_1,\dots,d_\ell,k_1,\dots,k_N\geq 0}\langle\kappa_1^{d_1}\dots\kappa_\ell^{d_\ell}\tau_{k_1}\dots\tau_{k_N}\rangle_{g,N}\right.$$
$$\left. t_{k_1}\cdots t_{k_N}\frac{s_1^{d_1}\dots s_\ell^{d_\ell}}{d_1!\cdots d_\ell!}\right),$$

where $\mathbf{s}$ denotes the infinite vector of independent variables $(s_1, s_2, \dots)$.

## Theorem 4.1 (Kaufmann–Manin–Zagier, Liu–Xu)

*The partition function* $Z^{\kappa}(\mathbf{t};\mathbf{s})$ *is a particular tau-function of the KdV hierarchy. Moreover,*

$$Z^{\kappa}(\mathbf{t};\mathbf{s}) = Z^{WK}(t_0, t_1, t_2 - h_1(-\mathbf{s}), t_3 - h_2(-\mathbf{s}), \ldots, t_{k+1} - h_k(-\mathbf{s}), \ldots)$$

*where* $h_k(\mathbf{s})$ *are polynomials in* $s_1, s_2, \ldots$ *defined through*

$$\sum_{k=0}^{\infty} h_k(\mathbf{s})\, x^k = \exp\left(\sum_{j=1}^{\infty} s_j x^j\right).$$

## Lemma 4.2

*The following formula holds true*

$$\exp\left(\sum_{m\geq 1} h_m(\mathbf{s})q_m\right) = \sum_{\lambda\in\mathbb{Y}} \frac{s_\lambda}{m(\lambda)!} \sum_{|\mu|=|\lambda|} L_{\lambda\mu}\, \frac{q_\mu}{m(\mu)!},$$

*where $L_{\lambda\mu}$ is the transition matrix from the monomial basis to the power sum basis. We have used the shorthand notations*

$$s_\lambda := \prod_{j=1}^{\ell(\lambda)} s_{\lambda_j}\ , \qquad q_\mu := \prod_{j=1}^{\ell(\mu)} q_{\mu_j}\,.$$

Now we are ready to give details of our results.

I. For a given $\lambda \in \mathbb{Y}$ and for any $N \geq 0$

$$\sum_{k_1,\ldots,k_N \geq 0} \langle \kappa_{\lambda_1} \ldots \kappa_{\lambda_{\ell(\lambda)}} \tau_{k_1} \ldots \tau_{k_N} \rangle \frac{(2k_1+1)!!}{z_1^{2k_1+2}} \cdots \frac{(2k_N+1)!!}{z_N^{2k_N+2}}$$

$$= (-1)^{\ell(\lambda)} \sum_{|\mu|=|\lambda|} \frac{L_{\lambda\mu}}{m(\mu)!} \operatorname*{res}_{w_1=\infty} \cdots \operatorname*{res}_{w_{\ell(\mu)}=\infty} dw_1 \cdots dw_{\ell(\mu)}$$

$$\frac{w_1^{2\mu_1+3}}{(2\mu_1+3)!!} \cdots \frac{w_{\ell(\mu)}^{2\mu_{\ell(\mu)}+3}}{(2\mu_{\ell(\mu)}+3)!!} F^{WK}_{\ell(\mu)+N}(w_1, \ldots, w_{\ell(\mu)}, z_1, \ldots, z_N).$$

II. The wave function of higher Weil–Petersson volumes satisfies

$$\psi^{\kappa}(z;\mathbf{t};\mathbf{s}) = \exp\left(\sum_{k=1}^{\infty}\frac{h_k(-\mathbf{s})z^{2k+3}}{(2k+3)!!}\right) \sum_{\lambda\in\mathbb{Y}}\frac{(-1)^{\ell(\lambda)}s_\lambda}{m(\lambda)!}\sum_{|\mu|=|\lambda|}L_{\lambda\mu}\frac{(-1)^{\ell(\mu)}}{m(\mu)!}\partial_{t_{\mu_1+1}}\dots\partial_{t_{\mu_{\ell(\mu)}+1}}\psi^{WK}(z;\mathbf{t}).$$

III. Denote

$$F^\kappa(z_1,\dots,z_N;\mathbf{s}) := \sum_{l\geq 0} \sum_{\substack{k_1,\dots,k_n\geq 0\\ d_1,\dots,d_l\geq 0}} \langle \kappa_1^{d_1}\dots\kappa_\ell^{d_\ell}\tau_{k_1}\dots\tau_{k_N}\rangle$$

$$\frac{(2k_1+1)!!}{z_1^{2k_1+2}}\cdots\frac{(2k_N+1)!!}{z_N^{2k_N+2}}\frac{s_1^{d_1}\dots s_\ell^{d_\ell}}{d_1!\cdots d_\ell!}, \quad N\geq 1.$$

Then we have

$$F^\kappa(z;\mathbf{s}) = \frac{-A(z)\,B_z(-z)+B_z(z)\,A(-z)+B(z)\,A_z(-z)-A_z(z)\,B(-z)}{4\,z},$$

$$F^\kappa(z_1,\dots,z_N;\mathbf{s}) = -\frac{1}{N}\sum_{r\in S_N}\frac{\mathrm{Tr}\,(M^\kappa(z_{r_1})\cdots M^\kappa(z_{r_N}))}{\prod_{j=1}^N(z_{r_j}^2-z_{r_{j+1}}^2)} - \delta_{N,2}\frac{z_1^2+z_2^2}{(z_1^2-z_2^2)^2}.$$

Here $A(z)=A(z;\mathbf{s}):=\psi^\kappa(z;0;\mathbf{s})$, $B(z)=B(z;\mathbf{s}):=\psi_x^\kappa(z;0;\mathbf{s})$, and

$$M^\kappa(z) = \frac{1}{2}\begin{pmatrix} -A(z)B(-z)-A(-z)B(z) & -2A(z)A(-z) \\ 2B(z)B(-z) & A(z)B(-z)+A(-z)B(z)\end{pmatrix}.$$

Note that $A(z)$, $B(z)$ are deformed Faber–Zagier series having the form

$$\begin{aligned} A(z) &= A^{WK}(z) + \sum_{k=1}^{\infty} \sum_{\lambda \in \mathbb{Y}_k} A^{\lambda}(z)\, s_{\lambda}, \\ B(z) &= B^{WK}(z) + \sum_{k=1}^{\infty} \sum_{\lambda \in \mathbb{Y}_k} B^{\lambda}(z)\, s_{\lambda}. \end{aligned}$$

## Example 4.3 (B.–D.–Y.)

We have

$$\langle \kappa_{3g-3} \rangle_{g,0} = \frac{1}{24^g \cdot g!}, \qquad g \geq 1,$$

$$\langle \kappa_1 \tau_{3g-3} \rangle_{g,1} = 3\frac{12g^2 - 12g + 5}{5!! \cdot 24^g \cdot g!}, \qquad g \geq 1,$$

$$\langle \kappa_2 \tau_{3g-4} \rangle_{g,1} = 3\frac{72g^3 - 132g^2 + 95g - 35}{7!! \cdot 24^g \cdot g!}, \qquad g \geq 2,$$

$$\langle \kappa_3 \tau_{3g-5} \rangle_{g,1} = \frac{1296g^4 - 3888g^3 + 4482g^2 - 2835g + 945}{9!! \cdot 24^g \cdot g!}, \qquad g \geq 2.$$

## Example 4.4 (B.–D.–Y.)

The first several terms of deformation of the Faber–Zagier series are

$$
\begin{aligned}
A^{(1)}(z) &= -\frac{z^5}{5!!}c + \frac{z^5}{5!!}q - \frac{z^2}{2\cdot 5!!}c,\\
A^{(2)}(z) &= -\frac{z^7}{7!!}c + \frac{8z^7+5z}{8\cdot 7!!}q - \frac{z^4}{2\cdot 7!!}c,\\
A^{(1^2)}(z) &= \left(\frac{z^{10}}{225} + \frac{11z^7}{1575} - \frac{z^4}{2520}\right)c + \left(-\frac{z^{10}}{225} - \frac{z^7}{210} + \frac{3z}{560}\right)q,\\
B^{(1)}(z) &= -\frac{z^6}{5!!}q + \frac{z^3}{2\cdot 5!!}q + \frac{4z^6-6}{4\cdot 5!!}c,\\
B^{(2)}(z) &= -\frac{z^8}{7!!}q + \frac{z^5}{2\cdot 7!!}q + \frac{8z^8-7z^2}{8\cdot 7!!}c,\\
B^{(1^2)}(z) &= \left(-\frac{z^{11}}{225} - \frac{z^8}{210} + \frac{z^5}{150} - \frac{z^2}{240}\right)c + \left(\frac{z^{11}}{225} + \frac{4z^8}{1575} - \frac{13z^5}{2520}\right)q.
\end{aligned}
$$

# 5. From simple Lie algebras to FJRW invariants

| $\mathfrak{g}$ | Dual fundamental series | No. of DFS |
| --- | --- | --- |
| $A_1$ | one exponential function | 1 |
| rk $2$ | Bessel functions | 2 |
| rk $3$ | two Bessel functions, one exponential | 3 |
| $A_4$ | solutions to scalar ODEs of order $4$ | 4 |
| $D_4$ | two Bessel functions, two exponential | 4 |

Table 1: The essential special functions in FJRW theory.

Dual fundamental series of $A_4$

$$\Big[21875\,\left(x^{12}+155\right)\,\theta^4-87500\,\left(7x^{12}+620\right)\,\theta^3$$
$$+4375\,\left(x^{24}+1277x^{12}+54870\right)\,\theta^2-8750\,\left(x^{24}+321x^{12}+31124\right)\,\theta$$
$$+7\left(x^{36}-1495x^{24}+510995x^{12}-9215525\right)\Big]\,\phi=0.$$

Here $\theta = x\frac{d}{dx}$.

**Thank you!**