Symmetries of CY hyper-surfaces in toric varieties

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Overview

Movitivation

- Freely-acting symmetries of Calabi-Yau manifolds play a crucial role in heterotic model building.
  Matter contents (representations of gauge bundles) define geometrical properties of CY. ($Ind(V) = -3$).
  Free quotient construction of simply connected CY. "Upstairs" & "Downstairs" manifolds. $X := \widetilde{X}/G$ [1] [2] [3]
- Kreuzer-Skarke list, [4] the ambient toric varieties correspond to (usually numerous) subdivisions of the normal fans of 473,800,776 reflexive 4-d polyhedron. [5] Only 16 of those lead to Calabi-Yau hypersurfaces with non-trivial fundamental group.
- Formally Volker had the classificaiton [6] on CICY. (Complete intersection CYs.)
- useful toolboxes: a. fixed point & smoothness check on toric CY. b. generalized Schur Cover. etc.

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[1] Yau, Shing-Tung. "Compact three dimensional Kahler manifolds with zero Ricci curvature."

[2] Tian, G., and Shing-Tung Yau. "Three-dimensional algebraic manifolds with c1= 0 and χ=-6.

[3] Greene, Brian R., et al. "A three-generation superstring model.

[4] Kreuzer, Maximilian, and Harald Skarke. "Complete classification of reflexive polyhedra in four dimensions." arXiv preprint hep-th/0002240 (2000).

[5] Batyrev, Victor, and Maximilian Kreuzer. "Integral cohomology and mirror symmetry for Calabi-Yau

Toric Variety

Definition: A **toric variety** is a complex variety X containing a torus $T := (\mathbb{C}^*)^r$ as an open dense subset, s.t. the action of T on itself extends to an action on all of X.

Ingredients:
$N_{\mathbb{R}}$ (for a toric 4-fold, 4-Dim lattice space.)
Charge Matrix $\mathcal{Q}$, the vertices of reflexive polytope.
Triangulation Σ.

the 1-cones/generators of Σ by $\mathcal{Q} := \Sigma^{(1)} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, where the $\mathbf{v}_i$ are d-dimensional vectors in $N_{\mathbb{R}}$

describe a set of generators $\{\mathbf{v}_{i_1}, \ldots \mathbf{v}_{i_p}\}$ which does not form a cone by the index set $I = \{i_1, \ldots, i_p\} \subset \{1, \ldots, n\}$ and associate to it a zero set

$$Z(I) = \{\mathbf{x} \in \mathbb{C}^n \,|\, x_i = 0 \;\; \forall\, i \in I\} \subset \mathbb{C}^n \,.$$

zero sets:

$$\mathcal{I} = \{I \subset \{1, \ldots, n\} \,|\, \{\mathbf{v}_i \,|\, i \in I\} \text{ do not form a cone}\} \,, \quad \mathcal{Z} = \{Z(I) \,|\, I \in \mathcal{I}\}$$

Toric Variety (cont.)

and the union of all zero sets by

$$Z = \bigcup_{I \in \mathcal{I}} Z(I)$$

then describe the toric action $\mathrm{T}^{n-d} \cong \mathcal{G} \cong (\mathbb{C}^*)^{n-d}$ as

$$\mathcal{G}_{\rm cont} = \{\mathrm{diag}(\mathbf{s}^{\mathbf{q}_i}) \,|\, \mathbf{s} \in (\mathbb{C}^*)^{n-d}\} \cong (\mathbb{C}^*)^{n-d} \ .$$

finally the toric variety (ambient space) is defined as:

$$A = \frac{\mathbb{C}^n - Z(\Sigma)}{\mathcal{G}} \ .$$

Example: $\mathbb{P}^2$

Toric Variety (cont.)

$$a = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, Z = \{x_1 = x_2 = x_3 = 0\}, \mathcal{G} = \mathrm{Ker}\{s_1, s_2, s_1^{-1}s_2^{-1}\} \cong \mathbb{C}^*$$

$$A = \mathbb{C}^3 - \{0,0,0\}/\mathbb{C}^* = (\mathbb{C}^*)^3/\mathbb{C}^* = \mathbb{P}^2$$

advantages:
1. Divisors 1-1 Vertices: $D_1 = \{x_1 = 0\}, D_2 = \{x_2 = 0\}, D_3 = \{x_3 = 0\}$
2. CY is easily computed. (Batyrev.)
   monomial 1-1 points
   $x \in N^\vee$ s.t. $\langle x, v\rangle > -1$

for $\mathbb{P}^2$:

$$\left\{x_3^3, x_2x_3^2, x_2^2x_3, x_2^3, x_1x_3^2, x_1x_2x_3, x_1x_2^2, x_1^2x_3, x_1^2x_2, x_1^3\right\}$$

.

G_A for $A = (\mathbb{C}^n - Z)/\mathcal{G}$

SR ideal defines zero loci:

$$Z(I) = \{\mathbf{x} \in \mathbb{C}^n \,|\, x_i = 0 \;\; \forall\, i \in I\} \subset \mathbb{C}^n \,.$$

$$\mathcal{I} = \{I \subset \{1, \ldots, n\} \,|\, \{\mathbf{v}_i \,|\, i \in I\} \text{ do not form a cone}\}$$

Define sub-group of S_n to define is the stabilizer group of $\mathcal{I}$,

$$S = \{\sigma \in S_n \,|\, \sigma(I) = I, \; \forall\, I \in \mathcal{I}\} \subset S_n \,,$$

which leaves all sets I in $\mathcal{I}$ invariant individually.

$$R = \{\sigma \in S_n \,|\, \sigma(I) \in \mathcal{I}, \; \forall I \in \mathcal{I}\} \subset S_n \,,$$

Define the quotient group

$$P = R/S$$

We proved the short exact group sequence

$$1 \to S \xrightarrow{\imath} R \xrightarrow{[\cdot]} P \to 1 \,.$$

where $\imath$ is the inclusion map and $[\cdot]$ denotes taking the equivalence class.

G_A for $A = (\mathbb{C}^n - Z)/\mathcal{G}$ (cont.)

Define an equivalence relation on $\{1, \dots, n\}$ by

$$i \sim j :\Leftrightarrow i \text{ and } j \text{ are contained in the same sets } I \in \mathcal{I} \, .$$

denote the equivalence classes by $\mathcal{J} = \{1, \dots, n\}/\sim$
every $I \in \mathcal{I}$ can be written as a disjoint union

$$I = \bigcup_{J \in \mathcal{J}, J \cap I \neq \emptyset} J \, .$$

The stabilizer group S is given by $S = \bigotimes_{J \in \mathcal{J}} S(J)$, where $S(J)$ is the group of permutations on the set J.
Moveover, R is a semi-direct product

$$R \cong P \ltimes S$$

Invariant group of Z.

Symmetries of $\mathbb{C}^n - Z$

the invariance group G_B, of the upstairs space $B = \mathbb{C}^n - Z$

from the definition of S, R, P it is clear that S_B, R_B, P_B leave the zero set Z invariant and are, hence, sup-groups of G_B.

Another obvious sub-group of G_B is

$$H_B = \{g \in G_B \,|\, g(Z(I)) = Z(I),\ \forall I \in \mathcal{I}\} \subset G_B\ ,$$

One can prove that G_B is a semi-direct product

$$G_B \cong P \ltimes H_B$$

, where only H_B remains to be computed. The answer:

$$G(\mathcal{J}) = \bigotimes_{J \in \mathcal{J}} \mathrm{Gl}(J, \mathbb{C})\ ,$$

and

$$G(\mathcal{J}) = H_B.$$

quotient by $\mathcal{G}$

Firstly compute the normalizer of G within Gl(n, C).
split the various coordinate directions up into disjoint blocks $\mathcal{K} = \{K \mid K \subset \{1, \ldots, n\}\}$, collecting the directions with the same charges $\mathbf{q}_i$, such that

$$\mathcal{G} = \bigotimes_{K \in \mathcal{K}} \mathcal{G}_{\mathbf{q}_K}(K)$$

where $\mathcal{G}_{\mathbf{q}_K}(K) = \{\mathbf{s}^{\mathbf{q}_K}\, \mathbf{1}_{|K|} \mid \mathbf{s} \in (\mathbb{C}^*)^{n-d}\}$

We would like to work out the normalizer

$$N_{\mathrm{Gl}(n,\mathbb{C})}(\mathcal{G}) = \{g \in \mathrm{Gl}(n,\mathbb{C}) \mid g\mathcal{G} = \mathcal{G}g\} = \{g \in \mathrm{Gl}(n,\mathbb{C}) \mid \forall \gamma \in \mathcal{G} \;\; \exists \tilde{\gamma} \in \mathcal{G} \,:\, g\gamma = \tilde{\gamma}g\}$$

It is also useful to introduce the commutant

$$C_{\mathrm{Gl}(n,\mathbb{C})}(\mathcal{G}) = \{g \in \mathrm{Gl}(n,\mathbb{C}) \mid g\gamma = \gamma g \;\forall \gamma \in \mathcal{G}\} \;,$$

Short answer:

$$N_{\mathrm{Gl}(n,\mathbb{C})}(\mathcal{G}) = \mathcal{P} \ltimes C_{\mathrm{Gl}(n,\mathbb{C})}(\mathcal{G}) \;,$$

where

$$\mathcal{P} = \mathcal{R}/\mathcal{S}$$

quotient by $\mathcal{G}$ (cont.)

$$\mathcal{R} = S_n \cap N_{\mathrm{Gl}(n,\mathbb{C})}(\mathcal{G}) \ , \quad \mathcal{S} = S_n \cap C_{\mathrm{Gl}(n,\mathbb{C})}(\mathcal{G}) \ ,$$

$$C_{\mathrm{Gl}(n,\mathbb{C})}(\mathcal{G}) = \bigotimes_{K \in \mathcal{K}} \mathrm{Gl}(K, \mathbb{C}) \ .$$

Next, compute the normalizer $N_{G_B}(\mathcal{G})$. Clearly, this normalizer is given by

$$N_{G_B}(\mathcal{G}) = G_B \cap N_{\mathrm{Gl}(n,\mathbb{C})}(\mathcal{G}) \ .$$

define the refined block-decomposition given as the intersection of $\mathcal{J}$ and $\mathcal{K}$ defined by

$$\mathcal{L} = \{L = J \cap K \,|\, J \in \mathcal{J} \, , \ K \in \mathcal{K} \, , \ L \neq \emptyset\} \ .$$

$$S_{\mathcal{L}} = \{\sigma \in S_n \,|\, \sigma(L) = L \ \ \forall L \in \mathcal{L}\} \ , \quad R_{\mathcal{L}} = \{\sigma \in S_n \,|\, \sigma(L) \in \mathcal{L} \ \ \forall L \in \mathcal{L}\} \ .$$

we can then form the quotient and the semi-direct product

$$P_{\mathcal{L}} = R_{\mathcal{L}} / S_{\mathcal{L}} \ , \quad R_{\mathcal{L}} = P_{\mathcal{L}} \ltimes S_{\mathcal{L}} \ .$$

from the definition of the various block structures is it clear that

$$H_A = H_B \cap C_{\mathrm{Gl}(n,\mathbb{C})}(\mathcal{G}) \ ,$$

quotient by $\mathcal{G}$ (cont.)

and we expect this to be the continuous part of $N_{G_B}(\mathcal{G})$. We also define the intersection of the permutation groups

$$R_A = R \cap \mathcal{R} \ , \quad S_A = S \cap \mathcal{S} \ , \quad P_A = R_A/S_A \ , \quad R_A = P_A \ltimes S_A \ .$$

It is now clear that we can write $N_{G_B}(\mathcal{G})$ as a semi-direct product

$$N_{G_B}(\mathcal{G}) = P_A \ltimes H_A$$

by dividing out the group S_A. To find the invariance group, G_A, of the toric space we have to divide this by $\mathcal{G}$ which results in

$$G_A = P_A \ltimes (H_A/\mathcal{G})$$

This completes the calculation of G_A.

Demazure roots.

The above can be elegantly rephrased in terms of Demazure roots.

$$\phi_{i\nu} : x_i \mapsto x_i + \lambda \prod_{k \neq i} x_k^{\langle \alpha_{i\nu}, v_k \rangle} ,$$

where $\alpha_{i\nu}$ are lattice points in M for which $\langle \alpha_{i\nu}, x_i \rangle = -1$ and $\langle \alpha_{i\nu}, x_k \rangle > -1$ for all $k \neq i$.
For any automorphism of the fan, i.e. maps in $GL(N)$ which preserve the set of cones in Σ', there is an associated permutation of homogeneous coordinates which gives rise to an automorphism of $\mathbb{P}_{\Sigma'}$.

Note that the Demazure roots, and hence the continuous part of the automorphism group, is completely independent of triangulation, as it only depends on the one-dimensional cones.

Twisted linear representations

A symmetry, Γ, of the Calabi-Yau hyper-surface $X \subset A$ is a sub-group of the ambient space symmetry group G_A, that is, there is a group monomomorphism $R : \Gamma \to G_A$. We can write $R(\gamma) = (\pi(\gamma), r(\gamma))$, where $\pi : \Gamma \to P_A$ and $r : \Gamma \to H_A/\mathcal{G}$.

From the multiplication rule $(p, h)(\tilde{p}, \tilde{h}) = (p\tilde{p}, \tilde{p}^1 h\tilde{p}\tilde{h})$ in G_A it follows that

$$\begin{aligned} \pi(\gamma\tilde{\gamma}) &= \pi(\gamma)\pi(\tilde{\gamma}) \\ r(\gamma\tilde{\gamma}) &= \pi(\tilde{\gamma})^{-1} r(\gamma)\pi(\tilde{\gamma})r(\tilde{\gamma})\ , \end{aligned}$$

so that $\pi : \Gamma \to P_A$ is a homomorphism and $r : \Gamma \to H_A/\mathcal{G}$ is a π-homomorphism.

assume that we have found a, not necessarily injective, group homomorphism

$$\pi : G \to P_A$$

Labelling the relevant blocks by B_i by $i = 1 \ldots k$, we can single out one of the blocks and consider its stabilizer $G_i \subset G$. Note that the restriction of $r(\gamma)$ for $\gamma \in G_i$ to B_i defines a representation of G_i. Let us call these representations

Twisted linear representations (cont.)

$r_i : G_i \to GL(B_i, \mathbb{C})$. An equivalent description is to write the representation $R(\gamma)$ as a product of two matrices

$$R(\gamma) = \pi(\gamma) \cdot \mathrm{diag}(r_1(\gamma), \cdots, r_n(\gamma))$$

Two steps.
linear projective representations $\to$ twisted linear representation.

Labelling the relevant blocks by B_i by $i = 1 \ldots k$, we can single out one of the blocks and consider its stabilizer $G_i \subset G$.
It is enough to fix a single of the r_i to recover the whole π-twisted representation by a splitting of G. Let us hence consider G_1, the stabilizer of the first block, and fix the representation $r_1 : G_1 \mapsto GL(B_1, \mathbb{C})$. To reconstruct the whole action of G we pick a set of group elements γ_i such that $\pi(\gamma_i)(1) = i$. We can make the choice $\gamma_1 = 1$ in G. We can then write any group element as

$$\gamma = \gamma_{\pi(\gamma)(i)} h \gamma_i^{-1} ,$$

with h in G_1, for any i. We can hence think of h to depend on γ and i. To see this, note that h is given

$$h = \gamma_{\pi(\gamma)(i)}^{-1} \gamma \gamma_i .$$

Twisted linear representations (cont.)

This is in G_1 as

$$\begin{aligned}\pi(h)(1) &= \pi(\gamma_{\pi(\gamma)(i)})^{-1}\pi(\gamma)\pi(\gamma_i)(1)\\ &= \pi(\gamma_{\pi(\gamma)(i)})^{-1}\pi(\gamma)(i)\\ &= 1\end{aligned}$$

We may hence write

$$R(\gamma) = R(\gamma_{\pi(\gamma)(i)})R(h)R(\gamma_i^{-1})$$

This allows us to recover all of the matrices $r_i(\gamma)$ and hence the entire π-twisted representation.

In general, it is not true, however, that $\pi : G \to P_A$ acts transitively on the blocks and there may be several orbits. In this case, we can do the above construction separately for each orbit O_k and then combine all of the data to find a linear representation $R : G \to P_A \ltimes H_A$

projective representations of G

$$\bar{r}:\Gamma \to \left[\prod_i GL(d_i,\mathbb{C})\right]/(\mathbb{C}^*)^{n-k}$$

where the j-th $\mathbb{C}^*$s acts as a matrix

$$C_j = \oplus \mathbb{I}_{d_i\times d_i}\lambda_j^{n_{j,i}}$$

on the homogeneous coordinates. The integers $n_{j,i}$ are the charges of the corresponding homogeneous coordinates.

In order to deal with a $\bar{r}$ we first need to recall a few facts about central extensions and Schur covers (see [Rotman], in particular Section 7, for details). First consider the central extension $\hat{\Gamma}_H$ of Γ_H which arises from the exact sequence

$$0 \to K \to \hat{\Gamma}_H \to \Gamma_H \to 0\ .$$

The extension $\hat{\Gamma}_H$ can be viewed as the set of pairs $(k,\gamma) \in K \times \Gamma_H$ with multiplication

$$(k,\gamma)(\tilde{k},\tilde{\gamma}) = (e(\gamma,\tilde{\gamma})k\tilde{k},\gamma\tilde{\gamma})\ ,$$

where $e:\Gamma_H \times \Gamma_H \to K$ is a factor set which defines a class $[e] \in H^2(\Gamma_H,K)$.

Smoothness of a symmetric CY

An hypersurface in $\mathbb{C}^n$ defined by a polynomial $P(x_1, \cdots, x_n)$, is smooth if there is no solution to

$$P = \frac{\partial P}{\partial x_1} = \cdots = \frac{\partial P}{\partial x_n} = 0$$

not directly applicable to toric varieties and we have to work using patches or look for an appropriate version.

$\mathbb{P}^n$: Method 1: We may cover $\mathbb{P}^n$ (with homogeneous coordinates $[x_1 : \cdots : x_{n+1}]$ with patches $\mathcal{U}_j : \{x_j \neq 0\}$ on which we have coordinates

$$\hat{x}_i = x_i / x_j \,.$$

In each patch we can examine the ideal, in the patch $\mathcal{U}_{n+1}$:

$$I_s^{\sigma_{n+1}} = \{P(\hat{x}_1, \cdots, \hat{x}_n) = \frac{\partial P}{\partial \hat{x}_1} = \cdots = \frac{\partial P}{\partial \hat{x}_n} = 0\}$$

Here, we do not go into patches but rather consider the ideal

$$I = \{P(x_1, \cdots, x_{n+1}) = \frac{\partial P}{\partial x_1} = \cdots = \frac{\partial P}{\partial x_n} = \frac{\partial P}{\partial x_{n+1}} = 0\}$$

Smoothness of a symmetric CY (cont.)

For a homogeneous hypersurface (of degree > 1) there is of course always the solution $x_i = 0 \forall i$, which is however not a point on $\mathbb{P}^n$. Let us hence consider the ideal

$$I_h = I - I_{SR}$$

where I_{SR} is the Stanley-Reisner ideal of $\mathbb{P}^n$. We now claim that $I_s = I_h$

we can associate a patch $V(\sigma$ to every four-dimensional cone σ_j in Σ'. Not all of these patches are given by $\mathbb{C}^4$ but can be $\mathbb{C}^4/G$ for a finite abelian group G The affine toric variety $V(\sigma) = (\mathbb{C}^*)^4$ associated with each smooth four-dimensional cone has a decomposition

$$V(\sigma) = (\mathbb{C}^*)^4 \amalg_{i=1..4} (\mathbb{C}^*)^3 \amalg_{j=1..6} (\mathbb{C}^*)^2 \amalg_{k=1..4} (\mathbb{C}^*) \amalg pt$$

in which each stratum $(C^*)^n$ is associated with an $4 - n$ dimensional cone contained in σ. we can use them as coordinates for the whole of $V(\sigma)$. These coordinates have the form

$$\hat{x}_i^\sigma \equiv \prod_j (x_j)^{\langle v_j, v_i^\nabla \rangle} = x_i^\sigma \prod_{\{x_j\} \setminus \{x_i^\sigma\}} x_j^{\langle v_j, v_i^\nabla \rangle}$$

Smoothness of a symmetric CY (cont.)

$$P_m = \prod x_j^{\langle v_j, m\rangle + 1} = \prod_j x_j \prod_i x_j^{m_i \langle v_j, v_i^\nabla \rangle}$$

$$= \prod_i (\hat{x}_i^\sigma)^{m_i} \left[\prod_j x_j \right]$$

$$= \prod_i (\hat{x}_i^\sigma)^{\langle v_i^\sigma, m\rangle + 1} \left[\prod_{\{x_j\} \setminus \{x_i^\sigma\}} x_j \right]$$

Not surprisingly, we can rewrite every monomial purely in terms of the $\hat{x}_i$ such that $P(x_i^\sigma) = P(\hat{x}_i^\sigma)$ when we set all other coordinates $= 1$

to check wether the hypersurface is smooth in a patch corresponding to a smooth cone we simply set all other coordinates to 1 and check that the ideal

$$I_s^\sigma = \{P^\sigma = \frac{\partial P^\sigma}{\partial x_1^\sigma} = \cdots = \frac{\partial P^\sigma}{\partial x_4^\sigma} = 0\}$$

Let us now discuss the patches which correspond to singular cones. Naively, singular patches arise because gauge fixing all coordinates except the ones in σ may leave some residual discrete group G in the $(\mathbb{C}^*)^{n-d}$ which we divide out. Hence the x_i^σ really parametrize the covering space and the patch is $\mathbb{C}^4/G$.

Smoothness of a symmetric CY (cont.)

Let us consider the example a Calabi-Yau (elliptic curve) hypersurface of $\mathbb{P}_{123}$. We take homogeneous coordinates $(y, x, z) \sim (\lambda^3 y, \lambda^2 x, \lambda z)$. It fan has rays generated by

$$v_y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad v_x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad v_z = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

The hypersurface can be described by

$$y^2 = x^3 + fxz^4 + gz^6$$

Let us discuss the cone σ spanned by x and z. Gauge fixing $y = 1$ leaves a residual $\mathbb{Z}_3$ acting on x and z so that we expect $V(\sigma) = \mathbb{C}^2/\mathbb{Z}_3$. The generators of σ^∇ are

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} \to \hat{z} = z^3/y \qquad \begin{pmatrix} -2 \\ 3 \end{pmatrix} \to \hat{x} = x^3/y^2$$

Clearly, we cannot rewrite (**??**) in terms of these coordinates alone, which corresponds to the fact that the above vectors do not generate the M-lattice. We can cure this by defining a new coordinate

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \to \hat{\xi} \equiv xz/y$$

Smoothness of a symmetric CY (cont.)

so the defining polynomial becomes (after dividing by y^2)

$$1 = \hat{x} + f\hat{\xi}\hat{z} + g\hat{z}^2 = 0$$

However, we now have the extra relation

$$\hat{\xi}^3 = \hat{z}\hat{x}$$

which is nothing but the defining equation of an A_3 singularity (which is the same as $\mathbb{C}^2/\mathbb{Z}_3$) embedded into $\mathbb{C}^3$.

we have to make sure our hypersurface X at $P = 0$ must not meet the singularities of the ambient space as this would lead to singularities on X as well. As the singularities arise through fixed points of the group action of G on $\mathbb{C}^4$ turning it into $V(\sigma) = \mathbb{C}^4/G$, we can lift $X \subset V(\sigma)$ to a (different) hypersurface $\tilde{X}$ in $\mathbb{C}^4$ which descends to X via a free group action of G. Smoothness of the covering space is then equivalent to smoothness of the quotient.

fixed points set of toric CY

To summarise the situation, we have an ambient space A and an upstairs version $B = \hat{A}$ such that $A = \hat{A}/\mathcal{G}$. The representation $R : \Gamma \to G_A$ acts on the ambient space A while its upstairs counterpart $\hat{R} : \hat{\Gamma} \to \hat{G}_A$, where $G_A = \hat{G}_A/\mathcal{G}$, acts on $\hat{A}$. I denote by $\Pi : \hat{\Gamma} \to \Gamma$ the group projection, by $\nu : \hat{G}_A \to G_A$ the projection for the ambient space symmetry groups and by $q : \hat{A} \to A$ the ambient space projection. Evidently, if $\Pi(\hat{\gamma}) = \gamma$ and $R(\gamma) = \nu \circ \hat{R}(\hat{\gamma})$ then

$$q \circ \hat{R}(\hat{\gamma}) = R(\gamma) \circ q \ .$$

For a $\gamma \in \Gamma$ define the associated downstairs fix point set by

$$F_\gamma = \{a \in A \mid R(\gamma)a = a\} \ ,$$

and its upstairs counterpart by

$$\hat{F}_{\hat{\gamma}} = \{\hat{a} \in \hat{A} \mid \hat{R}(\hat{\gamma})(\hat{a}) \in \hat{a}\mathcal{G}\} \ .$$