Hilbert schemes of points of the total space of $\mathcal{O}_{\mathbb{P}^1}(-n)$ as quiver varieties

Valeriano Lanza
valeriano@ime.unicamp.br

IMECC, University of Campinas (Brazil)

Workshop on Moduli Spaces in Algebraic Geometry and Mathematical Physics
Tianjin, 14-18 September 2015

UNICAMP

Hilbert schemes as quiver varieties
Valeriano Lanza
Monads for framed sheaves on Hirzebruch surfaces
ADHM data for Hilbert schemes
Hilbert schemes and quiver representations

Outline

1. Monads for framed sheaves on Hirzebruch surfaces
2. ADHM data for Hilbert schemes of points of Tot $\mathcal{O}_{\mathbb{P}^1}(-n)$
3. Hilbert schemes and quiver representations

Framed sheaves on Σ_n

Recall that $\Sigma_n = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)\right)$ is the nth Hirzebruch surface. We shall consider $n \geq 1$.
In the class $H \in \mathrm{Pic}(\Sigma_n)$, corresponding to the section of the ruling squaring to n, we fix a curve ℓ_∞ and call it the *line at infinity*.

Definition

A framed sheaf on Σ_n is a pair $(\mathcal{E}, \theta)$, where

- $\mathcal{E}$ is a torsion-free sheaf *trivial at infinity*, i.e.

$$\mathcal{E}|_{\ell_\infty} \simeq \mathcal{O}_{\ell_\infty}^{\oplus \operatorname{rk}(\mathcal{E})};$$

- $\theta \colon \mathcal{E}|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus \operatorname{rk}(\mathcal{E})}$ is an isomorphism.

Hilbert schemes as quiver varieties
Valeriano Lanza
Monads for framed sheaves on Hirzebruch surfaces
ADHM data for Hilbert schemes
Hilbert schemes and quiver representations

Framed sheaves on Σ_n

Recall that $\Sigma_n = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)\right)$ is the nth Hirzebruch surface. We shall consider $n \geq 1$.
In the class $H \in \mathrm{Pic}(\Sigma_n)$, corresponding to the section of the ruling squaring to n, we fix a curve ℓ_∞ and call it the *line at infinity*.

Definition

A framed sheaf on Σ_n is a pair $(\mathcal{E}, \theta)$, where

- $\mathcal{E}$ is a torsion-free sheaf *trivial at infinity*, i.e.

$$\mathcal{E}|_{\ell_\infty} \simeq \mathcal{O}_{\ell_\infty}^{\oplus \operatorname{rk}(\mathcal{E})} ;$$

- $\theta \colon \mathcal{E}|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus \operatorname{rk}(\mathcal{E})}$ is an isomorphism.

The theorem by Bartocci, Bruzzo, and Rava

There is a natural notion of isomorphism between framed sheaves.
Let $\mathcal{M}^n(r,a,c)$ be the set of isomorphism classes of framed sheaves on Σ_n having rank r, first Chern class aE, and second Chern class c, where $E \in \mathrm{Pic}(\Sigma_n)$ is the class of the unique irreducible curve with negative self-intersection.

Theorem (Bartocci, Bruzzo, Rava 2015)

The set $\mathcal{M}^n(r,a,c)$ is nonempty if and only if

$$c + \frac{1}{2}na(a-1) \geq 0\,.$$

If this is the case, it can be endowed with a structure of smooth algebraic variety, and turns out to be a fine moduli space for framed sheaves on Σ_n.

The theorem by Bartocci, Bruzzo, and Rava

There is a natural notion of isomorphism between framed sheaves.

Let $\mathcal{M}^n(r,a,c)$ be the set of isomorphism classes of framed sheaves on Σ_n having rank r, first Chern class aE, and second Chern class c, where $E \in \mathrm{Pic}(\Sigma_n)$ is the class of the unique irreducible curve with negative self-intersection.

Theorem (Bartocci, Bruzzo, Rava 2015)

The set $\mathcal{M}^n(r,a,c)$ is nonempty if and only if

$$c + \frac{1}{2}na(a-1) \geq 0\,.$$

If this is the case, it can be endowed with a structure of smooth algebraic variety, and turns out to be a fine moduli space for framed sheaves on Σ_n.

Idea of the proof

Key point

Any sheaf $\mathcal{E}$ trivial at infinity is isomorphic to the cohomology of a *monad*

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow 0\,, \tag{1}$$

where $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ are direct sums of line bundles depending only on the Chern invariants of $\mathcal{E}$.
We recall that a monad is a three-term complex which is exact in the first and last term, but may have cohomology in the middle.

The result is achieved by presenting $\mathcal{M}^n(r,a,c)$ as a quotient P/G, where P is a smooth algebraic variety, G is the automorphism group of (1), and the action is free.
The fineness is proved by introducing a suitable moduli functor and building up explicitly a universal family.

Idea of the proof

Key point

Any sheaf $\mathcal{E}$ trivial at infinity is isomorphic to the cohomology of a *monad*

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow 0\,, \tag{1}$$

where $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ are direct sums of line bundles depending only on the Chern invariants of $\mathcal{E}$.
We recall that a monad is a three-term complex which is exact in the first and last term, but may have cohomology in the middle.

The result is achieved by presenting $\mathcal{M}^n(r, a, c)$ as a quotient P/G, where P is a smooth algebraic variety, G is the automorphism group of (1), and the action is free.
The fineness is proved by introducing a suitable moduli functor and building up explicitly a universal family.

Idea of the proof

Key point

Any sheaf $\mathcal{E}$ trivial at infinity is isomorphic to the cohomology of a *monad*

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow 0\,, \qquad (1)$$

where $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ are direct sums of line bundles depending only on the Chern invariants of $\mathcal{E}$.
We recall that a monad is a three-term complex which is exact in the first and last term, but may have cohomology in the middle.

The result is achieved by presenting $\mathcal{M}^n(r, a, c)$ as a quotient P/G, where P is a smooth algebraic variety, G is the automorphism group of (1), and the action is free.
The fineness is proved by introducing a suitable moduli functor and building up explicitly a universal family.

The rank 1 case

- What are ADHM data?
- Which is the relation between framed sheaves on Σ_n and Hilbert schemes of points of $\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-n)$?

As for the second question, *the Hilbert scheme of c points of $\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-n)$ is isomorphic to the space $\mathcal{M}^n(1,0,c)$*.

The isomorphism is obtained by associating with a sheaf $\mathcal{E}$ the schematic support of $\mathcal{E}^{**}/\mathcal{E}$.
The triviality at infinity implies that this support is contained in $\Sigma_n \setminus \ell_\infty = \mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-n)$.

The rank 1 case

- What are ADHM data?
- Which is the relation between framed sheaves on Σ_n and Hilbert schemes of points of $\operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(-n)$?

As for the second question, *the Hilbert scheme of c points of $\operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(-n)$ is isomorphic to the space $\mathcal{M}^n(1,0,c)$.*

The isomorphism is obtained by associating with a sheaf $\mathcal{E}$ the schematic support of $\mathcal{E}^{**}/\mathcal{E}$.
The triviality at infinity implies that this support is contained in $\Sigma_n \setminus \ell_\infty = \operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(-n)$.

The rank 1 case

- What are ADHM data?
- Which is the relation between framed sheaves on Σ_n and Hilbert schemes of points of $\operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(-n)$?

As for the second question, *the Hilbert scheme of c points of $\operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(-n)$ is isomorphic to the space $\mathcal{M}^n(1,0,c)$.*

The isomorphism is obtained by associating with a sheaf $\mathcal{E}$ the schematic support of $\mathcal{E}^{**}/\mathcal{E}$.
The triviality at infinity implies that this support is contained in $\Sigma_n \setminus \ell_\infty = \operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(-n)$.

Hilbert schemes as quiver varieties

Valeriano Lanza

Monads for framed sheaves on Hirzebruch surfaces

ADHM data for Hilbert schemes

Hilbert schemes and quiver representations

The rank 1 case

- What are ADHM data?
- Which is the relation between framed sheaves on Σ_n and Hilbert schemes of points of $\operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(-n)$?

As for the second question, *the Hilbert scheme of c points of* $\operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(-n)$ *is isomorphic to the space* $\mathcal{M}^n(1,0,c)$.

The isomorphism is obtained by associating with a sheaf $\mathcal{E}$ the schematic support of $\mathcal{E}^{**}/\mathcal{E}$.
The triviality at infinity implies that this support is contained in $\Sigma_n \setminus \ell_\infty = \operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(-n)$.

What ADHM data are

ADHM data are a set of *linear data* satisfying some *closed and open conditions*, carrying an action of a group of matrices, which can be used to describe moduli spaces.

A significant example: Nakajima's triples

Nakajima proved that the Hilbert scheme of k points of $\mathbb{C}^2$ can be described by means of triples

$$(b_1, b_2, e) \in \mathrm{End}\left(\mathbb{C}^k\right) \oplus \mathrm{Hom}\left(\mathbb{C}^k, \mathbb{C}\right)$$

satisfying:

1. $[b_1, b_2] = 0$;
2. for all $(z, w) \in \mathbb{C}^2$ there is no nonzero vector $v \in \mathbb{C}^k$ s.t.

$$b_1 v = zv\,, \qquad b_2 v = wv\,, \qquad v \in \ker e\,.$$

What ADHM data are

ADHM data are a set of *linear data* satisfying some *closed and open conditions*, carrying an action of a group of matrices, which can be used to describe moduli spaces.

A significant example: Nakajima's triples

Nakajima proved that the Hilbert scheme of k points of $\mathbb{C}^2$ can be described by means of triples

$$(b_1, b_2, e) \in \mathrm{End}\left(\mathbb{C}^k\right) \oplus \mathrm{Hom}\left(\mathbb{C}^k, \mathbb{C}\right)$$

satisfying:

1. $[b_1, b_2] = 0$;
2. for all $(z, w) \in \mathbb{C}^2$ there is no nonzero vector $v \in \mathbb{C}^k$ s.t.

$$b_1 v = zv\,, \qquad b_2 v = wv\,, \qquad v \in \ker e\,.$$

We introduce the variety $P^n(c) \subset \mathrm{End}(\mathbb{C}^c)^{\oplus n+2} \oplus \mathrm{Hom}(\mathbb{C}^c, \mathbb{C})$, whose points $(A_1, A_2; C_1, \ldots, C_n; e)$ satisfy the conditions

Hilbert schemes as quiver varieties

Valeriano Lanza

Monads for framed sheaves on Hirzebruch surfaces

ADHM data for Hilbert schemes

Hilbert schemes and quiver representations

We introduce the variety $P^n(c) \subset \mathrm{End}(\mathbb{C}^c)^{\oplus n+2} \oplus \mathrm{Hom}(\mathbb{C}^c, \mathbb{C})$, whose points $(A_1, A_2; C_1, \ldots, C_n; e)$ satisfy the conditions

(P1)

$$A_1 C_1 A_2 = A_2 C_1 A_1 \qquad \text{when } n = 1\,,$$

$$\begin{cases} A_1 C_q = A_2 C_{q+1} \\ C_q A_1 = C_{q+1} A_2 \end{cases} \qquad q = 1, \ldots, n-1 \qquad \text{when } n \geq 2\,;$$

(P2) $A_1 + \lambda A_2$ is a *regular pencil* of matrices;

(P3) for all values of the parameters
$([\lambda_1, \lambda_2], (\mu_1, \mu_2)) \in \mathbb{P}^1 \times \mathbb{C}^2 \quad \text{s.t.} \quad \lambda_1^n \mu_1 + \lambda_2^n \mu_2 = 0,$
there is no nonzero vector $v \in \mathbb{C}^c$ such that

$$\begin{cases} C_1 A_2 v = -\mu_1 v \\ C_n A_1 v = (-1)^n \mu_2 v \\ v \in \ker e \end{cases} \quad \text{and} \quad (\lambda_2 A_1 + \lambda_1 A_2)\, v = 0\,.$$

Hilbert schemes as quiver varieties
Valeriano Lanza
Monads for framed sheaves on Hirzebruch surfaces
ADHM data for Hilbert schemes
Hilbert schemes and quiver representations

UNICAMP

We introduce the variety $P^n(c) \subset \mathrm{End}(\mathbb{C}^c)^{\oplus n+2} \oplus \mathrm{Hom}(\mathbb{C}^c, \mathbb{C})$, whose points $(A_1, A_2; C_1, \ldots, C_n; e)$ satisfy the conditions

(P1)

$$A_1 C_1 A_2 = A_2 C_1 A_1 \qquad \text{when } n = 1\,,$$

$$\begin{cases} A_1 C_q = A_2 C_{q+1} \\ C_q A_1 = C_{q+1} A_2 \end{cases} \qquad q = 1, \ldots, n-1 \qquad \text{when } n \geq 2\,;$$

(P2) $A_1 + \lambda A_2$ is a *regular pencil* of matrices;

(P3) for all values of the parameters $([\lambda_1, \lambda_2], (\mu_1, \mu_2)) \in \mathbb{P}^1 \times \mathbb{C}^2 \quad$ s.t. $\quad \lambda_1^n \mu_1 + \lambda_2^n \mu_2 = 0$, there is no nonzero vector $v \in \mathbb{C}^c$ such that

$$\begin{cases} C_1 A_2 v = -\mu_1 v \\ C_n A_1 v = (-1)^n \mu_2 v \\ v \in \ker e \end{cases} \quad \text{and} \quad (\lambda_2 A_1 + \lambda_1 A_2)\, v = 0\,.$$

Hilbert schemes as quiver varieties
Valeriano Lanza
Monads for framed sheaves on Hirzebruch surfaces
ADHM data for Hilbert schemes
Hilbert schemes and quiver representations

We introduce the variety $P^n(c) \subset \mathrm{End}(\mathbb{C}^c)^{\oplus n+2} \oplus \mathrm{Hom}(\mathbb{C}^c, \mathbb{C})$, whose points $(A_1, A_2; C_1, \dots, C_n; e)$ satisfy the conditions

(P1)

$$A_1 C_1 A_2 = A_2 C_1 A_1 \qquad \text{when } n = 1\,,$$

$$\begin{cases} A_1 C_q = A_2 C_{q+1} \\ C_q A_1 = C_{q+1} A_2 \end{cases} \qquad q = 1, \dots, n-1 \qquad \text{when } n \geq 2\,;$$

(P2) $A_1 + \lambda A_2$ is a *regular pencil* of matrices;

(P3) for all values of the parameters $([\lambda_1, \lambda_2], (\mu_1, \mu_2)) \in \mathbb{P}^1 \times \mathbb{C}^2$ s.t. $\lambda_1^n \mu_1 + \lambda_2^n \mu_2 = 0$, there is no nonzero vector $v \in \mathbb{C}^c$ such that

$$\begin{cases} C_1 A_2 v = -\mu_1 v \\ C_n A_1 v = (-1)^n \mu_2 v \\ v \in \ker e \end{cases} \qquad \text{and} \qquad (\lambda_2 A_1 + \lambda_1 A_2)\, v = 0\,.$$

Monads for framed sheaves on Hirzebruch surfaces

ADHM data for Hilbert schemes

Hilbert schemes and quiver representations

We consider the action of the group $\mathrm{GL}(c;\mathbb{C}) \times \mathrm{GL}(c;\mathbb{C})$ on $P^n(c)$ given by

$$(\phi_1, \phi_2) \cdot (A_i; C_j; e) = (\phi_2 A_i \phi_1^{-1}; \phi_1 C_j \phi_2^{-1}; e\phi_1^{-1}) .$$

Theorem (Bartocci, Bruzzo, L., Rava)

There is an isomorphism of smooth algebraic varieties

$$P^n(c)\,/\,\mathrm{GL}(c;\mathbb{C})^{\times 2} \simeq \mathcal{M}^n(1,0,c) = \mathrm{Hilb}^c(\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-n)) ,$$

and $P^n(c)$ is a locally trivial principal $\mathrm{GL}(c;\mathbb{C})^{\times 2}$-bundle over $\mathcal{M}^n(1,0,c)$.

Idea of the proof: we consider a suitable oper cover of $\mathrm{Hilb}^c(\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-n))$, whose elements are all isomorphic to $\mathrm{Hilb}^c(\mathbb{C}^2)$; then, we glue together Nakajima's triples.

We consider the action of the group $\mathrm{GL}(c;\mathbb{C}) \times \mathrm{GL}(c;\mathbb{C})$ on $P^n(c)$ given by

$$(\phi_1, \phi_2) \cdot (A_i; C_j; e) = (\phi_2 A_i \phi_1^{-1}; \phi_1 C_j \phi_2^{-1}; e\phi_1^{-1}) .$$

Theorem (Bartocci, Bruzzo, L., Rava)

There is an isomorphism of smooth algebraic varieties

$$P^n(c) \,/\mathrm{GL}(c;\mathbb{C})^{\times 2} \simeq \mathcal{M}^n(1,0,c) = \mathrm{Hilb}^c(\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-n)) ,$$

and $P^n(c)$ is a locally trivial principal $\mathrm{GL}(c;\mathbb{C})^{\times 2}$-bundle over $\mathcal{M}^n(1,0,c)$.

Idea of the proof: we consider a suitable oper cover of $\mathrm{Hilb}^c(\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-n))$, whose elements are all isomorphic to $\mathrm{Hilb}^c(\mathbb{C}^2)$; then, we glue together Nakajima's triples.

We consider the action of the group $\mathrm{GL}(c;\mathbb{C}) \times \mathrm{GL}(c;\mathbb{C})$ on $P^n(c)$ given by

$$(\phi_1, \phi_2) \cdot (A_i; C_j; e) = (\phi_2 A_i \phi_1^{-1}; \phi_1 C_j \phi_2^{-1}; e\phi_1^{-1}) .$$

Theorem (Bartocci, Bruzzo, L., Rava)

There is an isomorphism of smooth algebraic varieties

$$P^n(c) \,/ \mathrm{GL}(c;\mathbb{C})^{\times 2} \simeq \mathcal{M}^n(1,0,c) = \mathrm{Hilb}^c(\mathrm{Tot}\, \mathcal{O}_{\mathbb{P}^1}(-n)) ,$$

and $P^n(c)$ is a locally trivial principal $\mathrm{GL}(c;\mathbb{C})^{\times 2}$-bundle over $\mathcal{M}^n(1,0,c)$.

Idea of the proof: we consider a suitable oper cover of $\mathrm{Hilb}^c(\mathrm{Tot}\, \mathcal{O}_{\mathbb{P}^1}(-n))$, whose elements are all isomorphic to $\mathrm{Hilb}^c(\mathbb{C}^2)$; then, we glue together Nakajima's triples.

A quiver $\mathcal{Q}$ is a finite oriented graph, i.e. it is given by a (finite) set I of vertices and a (finite) set E of arrows.

A path in $\mathcal{Q}$ is given by a sequence of consecutive arrows, and the path algebra $\mathbb{C}\mathcal{Q}$ is the $\mathbb{C}$-algebra generated by all paths in $\mathcal{Q}$.

A representation of a quiver $\mathcal{Q}$ is given by the choice of a vector space for each vertex, and a linear map for each arrow.
An element $\mathbf{v} = (v_i)_{i \in I} \in \mathbb{N}^I$ is said to be a *dimension vector*, and a representation such that the vector space at the vertex i has dimension v_i is called $\mathbf{v}$*-dimensional*.

The space of all $\mathbf{v}$-dimensional representations of a quiver $\mathcal{Q}$ will be denoted by $\mathrm{Rep}(\mathcal{Q}, \mathbf{v})$.

A quiver $\mathcal{Q}$ is a finite oriented graph, i.e. it is given by a (finite) set I of vertices and a (finite) set E of arrows.

A path in $\mathcal{Q}$ is given by a sequence of consecutive arrows, and the path algebra $\mathbb{C}\mathcal{Q}$ is the $\mathbb{C}$-algebra generated by all paths in $\mathcal{Q}$.

A representation of a quiver $\mathcal{Q}$ is given by the choice of a vector space for each vertex, and a linear map for each arrow.
An element $\mathbf{v} = (v_i)_{i\in I} \in \mathbb{N}^I$ is said to be a *dimension vector*, and a representation such that the vector space at the vertex i has dimension v_i is called $\mathbf{v}$*-dimensional*.

The space of all $\mathbf{v}$-dimensional representations of a quiver $\mathcal{Q}$ will be denoted by $\mathrm{Rep}(\mathcal{Q}, \mathbf{v})$.

Hilbert schemes as quiver varieties
Valeriano Lanza
Monads for framed sheaves on Hirzebruch surfaces
ADHM data for Hilbert schemes
Hilbert schemes and quiver representations

A quiver $\mathcal{Q}$ is a finite oriented graph, i.e. it is given by a (finite) set I of vertices and a (finite) set E of arrows.

A path in $\mathcal{Q}$ is given by a sequence of consecutive arrows, and the path algebra $\mathbb{C}\mathcal{Q}$ is the $\mathbb{C}$-algebra generated by all paths in $\mathcal{Q}$.

A representation of a quiver $\mathcal{Q}$ is given by the choice of a vector space for each vertex, and a linear map for each arrow.
An element $\mathbf{v} = (v_i)_{i \in I} \in \mathbb{N}^I$ is said to be a *dimension vector*, and a representation such that the vector space at the vertex i has dimension v_i is called $\mathbf{v}$-*dimensional*.

The space of all $\mathbf{v}$-dimensional representations of a quiver $\mathcal{Q}$ will be denoted by $\mathrm{Rep}(\mathcal{Q}, \mathbf{v})$.

Hilbert schemes as quiver varieties
Valeriano Lanza
Monads for framed sheaves on Hirzebruch surfaces
ADHM data for Hilbert schemes
Hilbert schemes and quiver representations

A quiver $\mathcal{Q}$ is a finite oriented graph, i.e. it is given by a (finite) set I of vertices and a (finite) set E of arrows.

A path in $\mathcal{Q}$ is given by a sequence of consecutive arrows, and the path algebra $\mathbb{C}\mathcal{Q}$ is the $\mathbb{C}$-algebra generated by all paths in $\mathcal{Q}$.

A representation of a quiver $\mathcal{Q}$ is given by the choice of a vector space for each vertex, and a linear map for each arrow.
An element $\mathbf{v} = (v_i)_{i \in I} \in \mathbb{N}^I$ is said to be a *dimension vector*, and a representation such that the vector space at the vertex i has dimension v_i is called $\mathbf{v}$-*dimensional*.

The space of all $\mathbf{v}$-dimensional representations of a quiver $\mathcal{Q}$ will be denoted by $\mathrm{Rep}(\mathcal{Q}, \mathbf{v})$.

A quiver $\mathcal{Q}$ is a finite oriented graph, i.e. it is given by a (finite) set I of vertices and a (finite) set E of arrows.

A path in $\mathcal{Q}$ is given by a sequence of consecutive arrows, and the path algebra $\mathbb{C}\mathcal{Q}$ is the $\mathbb{C}$-algebra generated by all paths in $\mathcal{Q}$.

A representation of a quiver $\mathcal{Q}$ is given by the choice of a vector space for each vertex, and a linear map for each arrow.
An element $\mathbf{v} = (v_i)_{i \in I} \in \mathbb{N}^I$ is said to be a *dimension vector*, and a representation such that the vector space at the vertex i has dimension v_i is called $\mathbf{v}$-*dimensional*.

The space of all $\mathbf{v}$-dimensional representations of a quiver $\mathcal{Q}$ will be denoted by $\mathrm{Rep}(\mathcal{Q}, \mathbf{v})$.

There is a natural action of $G_{\mathbf{v}} := \prod_{i \in I} \mathrm{GL}(v_i; \mathbb{C})$ on $\mathrm{Rep}(\mathcal{Q}, \mathbf{v})$, given by change of basis.
One would like to "mod out" by this action.

1st approach $\longrightarrow$ to consider the orbit space
Issue $\longrightarrow$ bad behaviour (not even Hausdorff)

2nd approach $\longrightarrow$ affine algebro-geometric quotient
Issue $\longrightarrow$ massive loss of geometric information

Basically, one has two ways to fix the situation:

- recurring to *Geometric Invariant Theory (GIT)*;
- introducing a notion of *framing*.

There is a natural action of $G_{\mathbf{v}} := \prod_{i \in I} \mathrm{GL}(v_i; \mathbb{C})$ on $\mathrm{Rep}(\mathcal{Q}, \mathbf{v})$, given by change of basis.
One would like to "mod out" by this action.

1st approach	$\longrightarrow$	to consider the orbit space
Issue	$\longrightarrow$	bad behaviour (not even Hausdorff)
2nd approach	$\longrightarrow$	affine algebro-geometric quotient
Issue	$\longrightarrow$	massive loss of geometric information

Basically, one has two ways to fix the situation:

- recurring to *Geometric Invariant Theory (GIT)*;
- introducing a notion of *framing*.

There is a natural action of $G_{\mathbf{v}} := \prod_{i\in I} \mathrm{GL}(v_i;\mathbb{C})$ on $\mathrm{Rep}(\mathcal{Q},\mathbf{v})$, given by change of basis.
One would like to "mod out" by this action.

1st approach	$\longrightarrow$	to consider the orbit space
Issue	$\longrightarrow$	bad behaviour (not even Hausdorff)
2nd approach	$\longrightarrow$	affine algebro-geometric quotient
Issue	$\longrightarrow$	massive loss of geometric information

Basically, one has two ways to fix the situation:

- recurring to *Geometric Invariant Theory (GIT)*;
- introducing a notion of *framing*.

There is a natural action of $G_{\mathbf{v}} := \prod_{i \in I} \mathrm{GL}(v_i; \mathbb{C})$ on $\mathrm{Rep}(\mathcal{Q}, \mathbf{v})$, given by change of basis.
One would like to "mod out" by this action.

1st approach	$\longrightarrow$	to consider the orbit space
Issue	$\longrightarrow$	bad behaviour (not even Hausdorff)
2nd approach	$\longrightarrow$	affine algebro-geometric quotient
Issue	$\longrightarrow$	massive loss of geometric information

Basically, one has two ways to fix the situation:

- recurring to *Geometric Invariant Theory (GIT)*;
- introducing a notion of *framing*.

GIT prescribes:

- to introduce a notion of semistability, depending on the choice of a parameter $\vartheta \in \mathbb{Z}^I$; we replace $\mathrm{Rep}(Q, \mathrm{v})$ with the space of ϑ-semistable representations of Q, denoted by $\mathrm{Rep}^{ss}_{\vartheta}(Q, \mathrm{v})$;
- to change also the notion of quotient, according to the choice of the parameter ϑ; the GIT quotient will be denoted by $//_{\vartheta}$.

GIT prescribes:

- to introduce a notion of semistability, depending on the choice of a parameter $\vartheta \in \mathbb{Z}^I$; we replace $\mathrm{Rep}(\mathcal{Q}, \mathbf{v})$ with the space of ϑ-semistable representations of $\mathcal{Q}$, denoted by $\mathrm{Rep}^{ss}_{\vartheta}(\mathcal{Q}, \mathbf{v})$;
- to change also the notion of quotient, according to the choice of the parameter ϑ; the GIT quotient will be denoted by $/\!/_{\vartheta}$.

GIT prescribes:

- to introduce a notion of semistability, depending on the choice of a parameter $\vartheta \in \mathbb{Z}^I$; we replace $\mathrm{Rep}(\mathcal{Q}, \mathbf{v})$ with the space of ϑ-semistable representations of $\mathcal{Q}$, denoted by $\mathrm{Rep}^{ss}_{\vartheta}(\mathcal{Q}, \mathbf{v})$;
- to change also the notion of quotient, according to the choice of the parameter ϑ; the GIT quotient will be denoted by $//_{\vartheta}$.

Hilbert schemes as quiver varieties

Valeriano Lanza

Monads for framed sheaves on Hirzebruch surfaces

ADHM data for Hilbert schemes

Hilbert schemes and quiver representations

Toy example

If Q is the quiver

$$\overset{1}{\bullet} \xrightarrow{a} \overset{2}{\bullet} \xrightarrow{b} \overset{3}{\bullet} \quad ,$$

the quiver Q^{fr}, the framing of Q, is

$$\begin{array}{ccccc} \overset{1}{\bullet} & \xrightarrow{a} & \overset{2}{\bullet} & \xrightarrow{b} & \overset{3}{\bullet} \\ \downarrow j_1 & & \downarrow j_2 & & \downarrow j_3 \\ \underset{1'}{\bullet} & & \underset{2'}{\bullet} & & \underset{3'}{\bullet} \end{array} \quad .$$

WARNING: the change of basis is allowed only in the original vertices, not in the ones of framing.

Generalities on quivers
Framings

Hilbert schemes as quiver varieties

Valeriano Lanza

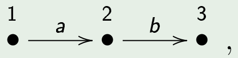
Monads for framed sheaves on Hirzebruch surfaces

ADHM data for Hilbert schemes

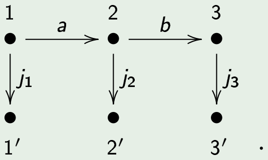
Hilbert schemes and quiver representations

Toy example

If $\mathcal{Q}$ is the quiver



the quiver $\mathcal{Q}^{\mathrm{fr}}$, the framing of $\mathcal{Q}$, is



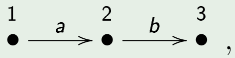
WARNING: the change of basis is allowed only in the original vertices, not in the ones of framing.

Generalities on quivers

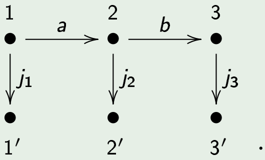
Framings

Toy example

If $\mathcal{Q}$ is the quiver



the quiver $\mathcal{Q}^{\mathrm{fr}}$, the framing of $\mathcal{Q}$, is



WARNING: the change of basis is allowed only in the original vertices, not in the ones of framing.

Hilbert schemes as quiver varieties

Valeriano Lanza

Monads for framed sheaves on Hirzebruch surfaces

ADHM data for Hilbert schemes

Hilbert schemes and quiver representations

The quivers $\mathcal{Q}_n$ and the ideals I_n

We introduce the quivers $\mathcal{Q}_n$

,

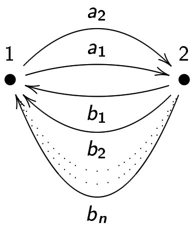
and the ideals $I_n \subset \mathbb{C}\mathcal{Q}_n$ generated by the relations

$$a_1 b_1 a_2 = a_2 b_1 a_1 \qquad \text{when } n = 1\,,$$

$$\begin{cases} a_1 b_q = a_2 b_{q+1} \\ b_q a_1 = b_{q+1} a_2 \end{cases} \qquad q = 1, \ldots, n-1 \qquad \text{when } n \geq 2\,.$$

The quivers $\mathcal{Q}_n$ and the ideals I_n

We introduce the quivers $\mathcal{Q}_n$



,

and the ideals $I_n \subset \mathbb{C}\mathcal{Q}_n$ generated by the relations

$$a_1 b_1 a_2 = a_2 b_1 a_1 \qquad \text{when } n = 1\,,$$

$$\begin{cases} a_1 b_q = a_2 b_{q+1} \\ b_q a_1 = b_{q+1} a_2 \end{cases} \qquad q = 1, \ldots, n-1 \qquad \text{when } n \geq 2\,.$$

Theorem (Bartocci, Bruzzo, L., Rava)

Let $B_n^{fr} = \mathbb{C}\mathcal{Q}_n^{fr}/I_n$, and fix $\mathbf{w} = (1, 0)$.
The variety $\mathrm{Hilb}^c(\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-n))$ is isomorphic to the quotient

$$\mathrm{Rep}^{ss}_{\vartheta_c}\left(B_n^{fr}, \mathbf{v}_c, \mathbf{w}\right) /\!/_{\vartheta_c} G_{\mathbf{v}_c} ,$$

where $\mathbf{v}_c = (c, c)$, and $\vartheta_c = (2c, -2c + 1)$.

Comments

- The dimension vector $\mathbf{w}$ refers to the vertices of framing.
- A representation of a quotient of the path algebra by an ideal I is a representation such that the linear maps satisfy the relations of I.

Hilbert schemes as quiver varieties

Valeriano Lanza

Monads for framed sheaves on Hirzebruch surfaces

ADHM data for Hilbert schemes

Hilbert schemes and quiver representations

The result

Theorem (Bartocci, Bruzzo, L., Rava)

Let $B_n^{fr} = \mathbb{C}Q_n^{fr}/I_n$, *and fix* $\mathbf{w} = (1,0)$.
The variety $\mathrm{Hilb}^c(\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-n))$ *is isomorphic to the quotient*

$$\mathrm{Rep}_{\vartheta_c}^{ss}\left(B_n^{fr}, \mathbf{v}_c, \mathbf{w}\right) //_{\vartheta_c} G_{\mathbf{v}_c}\,,$$

where $\mathbf{v}_c = (c,c)$, *and* $\vartheta_c = (2c, -2c+1)$.

Comments

- The dimension vector $\mathbf{w}$ refers to the vertices of framing.
- A representation of a quotient of the path algebra by an ideal I is a representation such that the linear maps satisfy the relations of I.

Hilbert schemes as quiver varieties

Valeriano Lanza

Monads for framed sheaves on Hirzebruch surfaces

ADHM data for Hilbert schemes

Hilbert schemes and quiver representations

The result

Theorem (Bartocci, Bruzzo, L., Rava)

Let $B_n^{fr} = \mathbb{C}\mathcal{Q}_n^{fr}/I_n$, *and fix* $\mathbf{w} = (1,0)$.
The variety $\operatorname{Hilb}^c(\operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(-n))$ *is isomorphic to the quotient*

$$\operatorname{Rep}^{ss}_{\vartheta_c}\left(B_n^{fr}, \mathbf{v}_c, \mathbf{w}\right) //_{\vartheta_c} G_{\mathbf{v}_c},$$

where $\mathbf{v}_c = (c,c)$, *and* $\vartheta_c = (2c, -2c+1)$.

Comments

- The dimension vector $\mathbf{w}$ refers to the vertices of framing.
- A representation of a quotient of the path algebra by an ideal I is a representation such that the linear maps satisfy the relations of I.

Further developments

- The total space of $\mathcal{O}_{\mathbb{P}^1}(-2)$ is symplectic, and so the Hilbert scheme. This property has as a countepart in the quiver setting the fact that $\mathcal{Q}_2$ is a double $\longrightarrow$ Nakajima quiver varieties.
  For $n \neq 2$ our Hilbert schemes are not symplectic, but they are Poisson: *is it possible to read the Poisson structures direclty from the quiver, as it happens in the symplectic case?*
- Find ADHM data (and/or a quiver description) for all Chern invariants.

Further developments

- The total space of $\mathcal{O}_{\mathbb{P}^1}(-2)$ is symplectic, and so the Hilbert scheme. This property has as a countepart in the quiver setting the fact that $\mathcal{Q}_2$ is a double $\longrightarrow$ Nakajima quiver varieties.
  For $n \neq 2$ our Hilbert schemes are not symplectic, but they are Poisson: *is it possible to read the Poisson structures direclty from the quiver, as it happens in the symplectic case?*
- Find ADHM data (and/or a quiver description) for all Chern invariants.

- The total space of $\mathcal{O}_{\mathbb{P}^1}(-2)$ is symplectic, and so the Hilbert scheme. This property has as a countepart in the quiver setting the fact that $\mathcal{Q}_2$ is a double
$\longrightarrow$ Nakajima quiver varieties.
For $n \neq 2$ our Hilbert schemes are not symplectic, but they are Poisson: *is it possible to read the Poisson structures direclty from the quiver, as it happens in the symplectic case?*
- Find ADHM data (and/or a quiver description) for all Chern invariants.

Thanks for your attention!