Elliptically fibred CY 3-folds and the ring of weak Jacobi forms

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❶ Topological String Theory:

Classically string theory is defined by map

$$x : \Sigma_g \to M \times \mathbb{R}_{3,1}$$

from a 2d world-sheet Σ_g of genus g into a target space $M \times \mathbb{R}_{3,1}$. Σ_g is equipped with a 2d super diffeomorphism invariant action S, of type II. The partition function of the first quantized string is formally

$$Z(G, B) = \int \mathcal{D}x\mathcal{D}h\mathcal{D}\mathrm{ferm}\ e^{\frac{i}{\hbar}S(x,h,\phi,\mathrm{ferm},G,B)},$$

with the bosonic part of the action ($\sim$ area$_{\mathbb{C}}$).

$$S_B(x,h,\phi,M) = \frac{2\pi}{\alpha'}\int_{\Sigma_g} \mathrm{d}\sigma^2 \partial_\alpha x^i \partial_\beta x^j (h^{\alpha\beta}\sqrt{h}G_{ij} + i\epsilon^{\alpha\beta}B_{ij}) \\ -\frac{1}{2\pi}\phi\int_{\Sigma_g} R^{(2)} \quad \leftarrow \textit{topol. term } \phi(2-2g)$$

where metric G and B field on $M \times \mathbb{R}_{3,1}$ are background parameters.

- For $d_{crit} = 10$ the variational integral $\int \mathcal{D}h$ over the w-s metric becomes a discrete integral $\sum_g \int_{\mathcal{M}_g} d\mu_g$ over the $3g-3$ dimensional moduli space of cmpl. str. on Σ_g.

- If M is a Calabi-Yau 3-fold $\exists\,(J_{11},\Omega_{30})$ & $c_1(TM)=0$
  - ➪ S has $(2,2)$ super conformal symmetry with four nilpotent operators $Q^{\pm},\bar{Q}^{\pm}$. Vector- and axial $U(1)$ allow to define twisted nilpotent scalar operators Q_A and Q_B, which define two inequivalent cohomological topological string theories, called A and B model.
  - ➪ In the A-model, super symmetric localisation localizes to $\delta S_B=0$, i.e. maps with minimal area called G (j,J) holomophic maps x_{hol}, so that

$$\int \mathcal{D}x\mathcal{D}h \rightarrow \sum_{g,\beta\in H_2(M,\mathbb{Z})} \int_{\mathcal{M}_{g,\beta}} c^{vir}_{g,\beta} = \sum_{g,\beta\in H_2(M,\mathbb{Z})} r^{\beta}_{g}$$

localises to a sum over finite dimensional

$$\dim_{\mathbb{C}}(\mathcal{M}_{g,\beta}) = \int_{\mathcal{C}_\beta} c_1(T_M) + (\dim(M) - 3)(1-g)$$

integrals over the moduli space of $x_{hol} : \Sigma_g \to [\mathcal{C}_\beta] \in M$. The latter are mathematically defined sympletic invariants called Gromow-Witten invariants. We get a mathematical definition of the topol. sector of string theory in a large $vol = \int_{[\mathcal{C}_a]} J$ expansion

$$Z = \exp\left(\sum_{g=0}^{\infty} {g_s}^{2g-2} F_g(\underline{z(\underline{t})})\right), \quad F_g = \sum_{\beta \in H_2(M,\mathbb{Z})} r_g^\beta Q^\beta$$

with $Q^\beta = \exp(2\pi i \sum_a t_a \beta^a)$ and $t_a = \int_{[\mathcal{C}_a]}(B + iJ)$. Here $r_g^\beta \in \mathbb{Q}$ are the Gromov-Witten invariants. While each coefficient of $F = \log(Z)$ is mathematically well defined and $F_g(Q)$ is a convergent series in Q for $|Q| < c$, the sum over g yields ony an asymptotic expansion for all values of Q. E.g. for $Q = 0$ the constant map contribution to each genus is for $g > 1$

$$F_g(0) = (-1)^g \frac{\chi(M)}{2} \int_{\overline{\mathcal{M}}_g} c_{g-1}^3 = (-1)^g \frac{\chi}{2} \frac{|B_{2g} B_{2g-2}|}{(2g(2g-2)!} \, .$$

This is a divergent sum because of the growth of the Bernoulli numbers B_m.

➪ After resumation and motivic refinement $F(g_s, t)$ has an even more interesting intepretation as generation function for the dimensions $N^{\beta}_{j_l j_r}$ of vector spaces of BPS representations.

$$F(\epsilon_1, \epsilon_2, t) = \sum_{\substack{j_l, j_r \\ m \geq 1 \beta \in H_2}} \frac{N^{\beta}_{j_l j_r}}{m} \frac{[j_l]_s [j_r]_r Q^{\beta m}}{(q_1^{\frac{m}{2}} - q_1^{-\frac{m}{2}})(q_2^{\frac{m}{2}} - q_2^{-\frac{m}{2}})}$$

Here $j_l, j_r \in \frac{1}{2}\mathbb{N}$ are spin representations of the BPS state and we define $[j]_x = \frac{x^{2j-1} - x^{-2j-1}}{x - x^{-1}}$ and $q_k = e^{2\pi i \epsilon_k}$, $k = 1, 2$, $s = \sqrt{\frac{q_1}{q_2}}$ and $r = \sqrt{q_1 q_2}$ for the parameters labelling the spin content.

Mathematically $N^{\beta}_{j_l j_r} \in \mathbb{N}$ is the dimension of vectors spaces that correspond to an $sl_l(2) \times sl_r(2)$ Lefshetz decompostion of the cohomology of the moduli of stable pairs. Choi, Katz, AK : 1210.4403.

Example 1: Huang, Poretschkin, AK: 1308.0619 local del Pezzo Calabi-Yau space $\mathcal{O}(-K_S) \to S$ with $S = d_8\mathbb{P}^2$.

For the BPS states $N^{d_K}_{j_l, j_r}$ at $d_K = 2$ one gets:

$2j_l \backslash 2j_r$	0	1	2	3
0		3876		
1			248	
2				1

$$d_K = 2$$

It is obvious that the adjoint represention $\mathbf{248}$ of

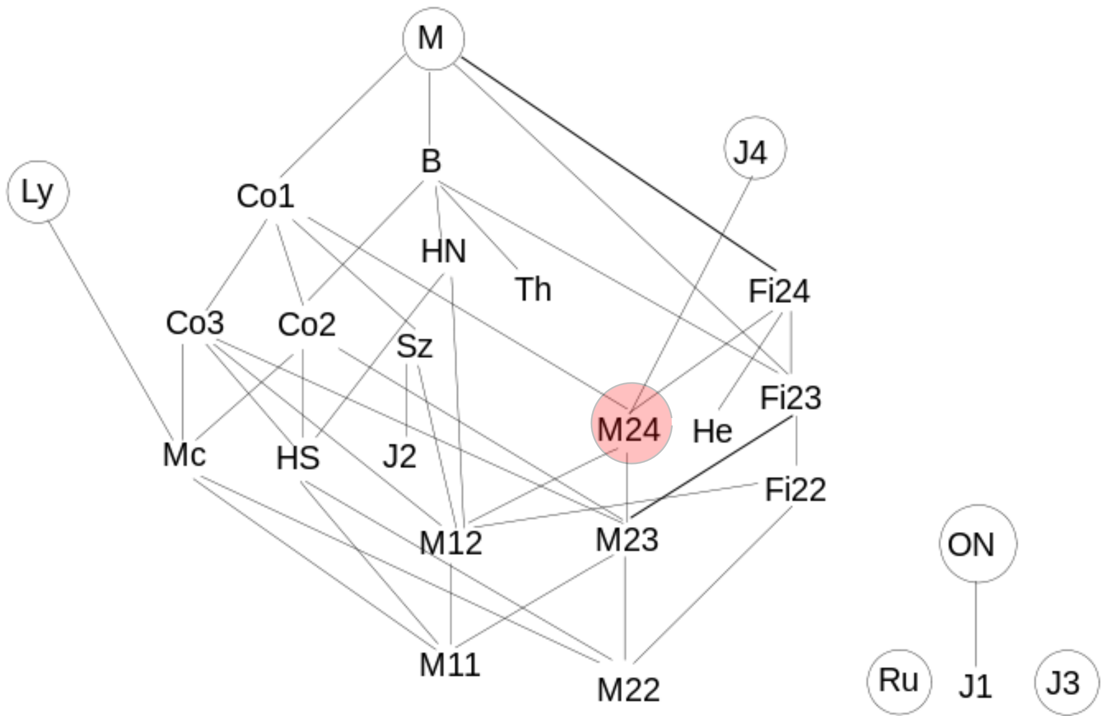
E_8 appears as the spin $N^1_{\frac{1}{2},\frac{3}{2}}$, which decomposes into two Weyl orbits with the weights $w_1 + 8w_0$, further $3876 = \mathbf{1} + \mathbf{3875}$, where the latter decomposes in the Weyl orbits of $w_1 + 7w_8 + 35w_0$.

Example 2: Katz, Pandharipande, AK: 1407.3181 $S = K3$ For the BPS states $N^d_{j_l,j_r}$ at $d = 3$ one gets:

$2j_L \backslash 2j_R$	0	1	2	3
0	1981		1	
1		252		
2	1		21	
3				1

$d = 3$

Now $1981 = 2 \cdot \mathbf{990} + \mathbf{1}$ and $\mathbf{252}$ are representations of the Mathieu group $M_{24} \in S_{24}$, which is one of sporadic finite groups



of order $|\mathbb{M}_{24}| = 244823040$.

➪ The topological string limit of the refined theory is

$\epsilon_1 = -\epsilon_2 = \frac{g_s}{2\pi}$. In this limit

$$F^{TS}(g_s, t) = \sum_{\substack{g \geq 0 \\ m \geq 1 \beta \in H_2}} \frac{n_g^\beta}{m} \left(2 \sin \frac{m g_s}{2}\right)^{2g-2} Q^{\beta m} \; .$$

here $n_\beta^g \in \mathbb{Z}$ are the Gopakumar-Vafa invariants. The are indices in the Hilbert space of BPS invariants, which are invariant under complex deformations of M. Note that $F^{TS}(g_s, t)$ has poles at

$$g_s = 2\pi \mathbb{Q}$$

from the genus zero sector.

➪ Z captures in particular integer symplectic invariants, the DT- or the PT- invariants, physically related to the unrefined BPS invariants $n_g^\beta \in \mathbb{Z}$ as can be seen from the product formula

$$Z = \prod_\beta \left[\left(\prod_{m=k}^{\infty} (1 - y^k q^\beta)^{k n_0^\beta} \right) \prod_{\substack{g=1 \\ l=0}}^{\substack{2g-2 \\ \infty}} (1 - y^{g-l-1} Q^\beta)^{(-1)^{g+l} \binom{2g-2}{l} n_g^\beta} \right],$$

where $y = \exp(g_s)$.

➪ For non-compact toric Calabi-Yau spaces $\mathcal{O}(-K_S) \to S$ the invararariants can be calculated using localization, the topological vertex, Matrix model techniques. The real challenge is for compact Calabi-Yau spaces

➪ **Physics:** These geometric invariants determine parts of the spectrum of string, M- and F-theory compactifications. A direct physical motivation is to calculate the BPS saturated correlations functions in the effective 4d (6d) $N=2$ field theory $F := F_0$

➪ gauge coupl: $g_{IJ}^{-2} = \mathrm{Im}\left(\bar{F}_{IJ} + \frac{2i\mathrm{Im}F_{IK}\mathrm{Im}F_{IL}X^K X^L}{\mathrm{Im}F_{KL}X^K X^L}\right)$

➪ BPS masses: $M^2_{n_E,n_M} = e^K \left|n_E t_E + n_M F_M\right|^2$

➪ grav couplings: $\int_{\mathrm{d}} x^4 F^g(t,\bar{t}) F_+^{2g-2} R_+^2$.

❷ The result:

Let M be an elliptically fibred 3-fold over a 2d fano surface S_2

$$\mathcal{E} \longrightarrow M \longrightarrow S_2$$

To be concrete we consider here the simplest case $S_2 = \mathbb{P}^2$ and call τ and t_B be the Kähler parameters of the elliptic fiber $\mathcal{E}$ and a line $l \subset \mathbb{P}^2$, $q = \exp(2\pi i \tau)$ and $Q = \exp(2\pi i t_B)$. We expand $\mathcal{Z}$ in terms of the base degrees d_B as

$$Z(\underline{t}, g_s) = \mathcal{Z}_0(\tau, \lambda) \left(1 + \sum_{d_B=1}^{\infty} Z_{d_B}(\tau, g_s) Q^{d_B} \right) \quad .$$

Then $Z_{d_B>0}$ is a quotient of even weak Jacobi forms of the following form HKK'15

$$Z_{d_B}(\tau,z) = \frac{\varphi_{d_B}(\tau,z)}{\eta^{36 d_B}(\tau)\prod_{k=1}^{d_B}\varphi_{-2,1}(\tau,kz)}. \qquad (1)$$

Here $\eta(\tau)$ is the Dedekind function and $\varphi_{d_B}(\tau,z)$ is an even weak Jacobi form of index $\frac{1}{3}d_B(d_B-1)(d_B+4)$ and weight $16d_B$.

☆ Jacobi forms

❶ Definition of Jacobi forms

Jacobi forms $\varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ depend on a modular parameter $\tau \in \mathbb{H}$ and an elliptic parameter $z \in \mathbb{C}$. They transform under the modular group (Eichler & Zagier)

$$\tau \mapsto \tau_\gamma = \frac{a\tau + b}{c\tau + d}, \; z \mapsto z_\gamma = \frac{z}{c\tau + d} \text{ with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z}) =: \Gamma_0$$

as

$$\varphi\left(\tau_\gamma, z_\gamma\right) = (c\tau + d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} \varphi(\tau, z)$$

and under quasi periodicity in the elliptic parameter as

$$\varphi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m(\lambda^2\tau+2\lambda z)}\varphi(\tau,z), \quad \forall \quad \lambda,\mu \in \mathbb{Z}\ .$$

Here $k \in \mathbb{Z}$ is called the *weight* and $Bm \in \mathbb{Z}_{>0}$ is called the *index* of the Jacobi form.

The Jacobi forms have a Fourier expansion

$$\phi(\tau,z) = \sum_{n,r} c(n,r) q^n y^r, \qquad \text{where } q = e^{2\pi i\tau},\ \ y = e^{2\pi i z}$$

Because of the quasi peridicity one has $c(n,r) =: C(4nm - r^2, r)$, which depends on r only modulo $2m$. For a holomorphic Jacobi form $c(n, r) = 0$ unless $4mn \geq r^2$, for cusp forms $c(n, r) = 0$ unless $4mn > r^2$, while for weak Jacobi forms one has only the condition $c(n, r) = 0$ unless $n \geq 0$.

❷ The ring of weak Jacobi forms

A weak Jacobi form of given index m and even modular weight k is freely generated over the ring of modular forms of level one, i.e. polynomials in $Q = E_4(\tau)$, $R = E_6(\tau)$ and $A = \varphi_{0,1}(\tau, z)$, $B = \varphi_{-2,1}(\tau, z)$ as

$$J^{weak}_{k,m} = \bigoplus_{j=0}^{m} M_{k+2j}(\Gamma_0)\varphi^{j}_{-2,1}\varphi^{m-j}_{0,1} \ .$$

The generators are the Eisenstein series E_4, E_6

$$E_k(\tau) = \frac{1}{2\zeta(k)} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k} = 1 + \frac{(2\pi i)^k}{(k-1)!\zeta(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \ ,$$

as well as

$$A = -\frac{\theta_1(\tau,z)^2}{\eta^6(\tau)}, \qquad B = 4\left(\frac{\theta_2(\tau,z)^2}{\theta_2(0,\tau)^2} + \frac{\theta_3(\tau,z)^2}{\theta_3(0,\tau)^2} + \frac{\theta_4(\tau,z)^2}{\theta_4(0,\tau)^2}\right) \ .$$

To summarize generators and quantitites defining the tpological string partition function

	Q	R	A	B	φ_{d_B}	$Z_{d_B}(\tau,z)$
weight k:	4	6	-2	0	$16d_B$	0
index m:	0	0	1	1	$\frac{1}{3}d_B(d_B-1)(d_B+4)$	$\frac{d_B(d_B-3)}{2}$

Since the numerator in

$$Z_{d_B}(\tau,z) = \frac{\varphi_{d_B}(\tau,z)}{\eta^{36 d_B}(\tau)\prod_{k=1}^{d_B}\varphi_{-2,1}(\tau,kz)}.$$

is finitly generated, we can get for each d_B the full genus answer based on a finite number of data.

Using as boundary data

- the conifold gap condition. Huang, Quakenbush, A.K. hep-th/0612125

- the involution symmetry on $\mathcal{M}$ $I:\Omega\mapsto i\Omega \leftrightarrow$ fibre modularity

- the parametrization of Z in terms of weak Jacobi-Forms

we can solve the compact elliptic fibration over $\mathbb{P}^2$ to

$d_B = 20$ for all d_E and all genus or to genus 189 for all classes d_B and d_E.

❸ Witten's wave function and weak Jacobi-forms

Witten gave a wave function interpretation the topological string partition function, which implies

$$\left(\frac{\partial}{\partial (t')^{\bar a}}+\frac{i}{2}\, R g_s^2 C_{\bar a\bar b\bar c} g^{b\bar b} g^{c\bar c}\frac{D}{Dt^b}\frac{D}{Dt^c}\right) Z(\lambda,\tau,t_B)=0\ ,$$

and summarizes all holomorphic anomaly equations

If we apply this equation to Z with $(t')^{\bar a}=\bar\tau$ and $Q^\beta=e^{2\pi i d_B t_B}$, we get in the large base because of the special from of the intersection matrix of elliptically fibered Calabi-Yau 3 folds only derivatives in the base direction t_B for t^b and t^c. Identifying g_s with $2\pi iz$ this becomes

$$\left(\partial_{\hat{E}_2} + \frac{d_B(d_B-3)}{24}z^2\right) \mathcal{Z}_{d_B}(\tau, z) = 0,$$

which is solved by a weak Jacobi form of index $m = \frac{d_B(d_B-3)}{2}$.

Because of modularity and quasiperiodicity given a weak Jacobi form $\varphi_{k,m}(\tau, z)$ one can always define modular form of weight k as follows

$$\tilde{\varphi}_k(\tau, z) = e^{\frac{\pi^2}{3} m z^2 E_2(\tau)} \varphi_{k,m}(\tau, z) \ .$$

It follows that the weak Jacobi forms $\varphi_{k,m}(\tau, z)$ have a Taylor expansion in z with coefficients that are quasi-modular forms as in Eichler and Zagier [1].

$$\varphi_{k,m} = \xi_0(\tau) + \left(\frac{\xi_0(\tau)}{2} + \frac{m\xi_0'(\tau)}{k} \right) z^2 + \left(\frac{\xi_2(\tau)}{24} + \frac{m\xi_1'(\tau)}{2(k+2)} + \frac{m^2\xi_0''(\tau)}{2k(k+1)} \right) z^4 + \mathcal{O}(z^6) \ .$$

Moreover one has

$$\left(\partial_{E_2} + \frac{m\lambda^2}{12} \right) \varphi_{k,m}(\tau, z) = 0 \ .$$

[1] E.g. $\phi_{-2,1}(\tau, z) = -z^2 + \frac{E_2 z^4}{12} + \frac{-5E_2^2 + E_4}{1440} z^6 + \frac{35E_2^3 - 21E_2E_4 + 4E_6}{362880} z^8 + \mathcal{O}(z^{10})$.

In particular A and B are quasi-modular forms that satisfy the modular anomaly equation

$$\partial_{E_2} A = -\frac{\lambda^2}{12} A, \qquad \partial_{E_2} B = -\frac{\lambda^2}{12} B \ . \tag{2}$$

We can write this as the holomorphic anomaly equation

$$\left(2\pi i \mathrm{Im}^2(\tau)\bar{\partial}_{\bar{\tau}} - \frac{m\lambda^2}{4}\right) \hat{\varphi}_{k,m}(\tau, z) = 0 \ . \tag{3}$$

☆ Elliptically fibred CY- manifolds

❶ Global fibration over $\mathbb{P}^2$

The formalism leads to a series of all genus predictions of BPS invariants for low base degree HKK'15. E.g. for $d_b = 1$ and $d_b = 2$ the numerator is

$$\varphi_1 = -\frac{Q(31Q^3 + 113P^2)}{48},$$

which leads to the following prediction of BPS invariants

$g\backslash d_E$	$d_E=0$	1	2	3	4	5	6
$g=0$	3	-1080	143370	204071184	21772947555	1076518252152	33381348217290
1	0	-6	2142	-280284	-408993990	-44771454090	-2285308753398
2	0	0	9	-3192	412965	614459160	68590330119
3	0	0	0	-12	4230	-541440	-820457286
4	0	0	0	0	15	-5256	665745
5	0	0	0	0	0	-18	6270
6	0	0	0	0	0	0	21

Table 1: Some BPS invariants $n^g_{(d_E,1)}$ for base degree $d_B=1$ and $g,d_E\le 6$.

$$\begin{aligned}\varphi_2 \quad = \quad & \frac{B^4Q^2\left(31Q^3+113R^2\right)^2}{23887872}+\frac{1}{1146617856}[2507892B^3AQ^7R+9070872B^3AQ^4R^3 \\ & +2355828B^3AQR^5+36469B^2A^2Q^9+764613B^2A^2Q^6R^2-823017B^2A^2Q^3R^4 \\ & +21935B^2A^2R^6-9004644BA^3Q^8R-30250296BA^3Q^5R^3-6530148BA^3Q^2R^5 \\ & +31A^4Q^{10}+5986623A^4Q^7R^2+19960101A^4Q^4R^4+4908413A^4QR^6]\ , \end{aligned} \tag{4}$$

$g\backslash d_E$	$d_E = 0$	1	2	3	4	5	6
$g = 0$	6	2700	-574560	74810520	-49933059660	7772494870800	31128163315047072
1	0	15	-8574	2126358	521856996	1122213103092	879831736511916
2	0	0	-36	20826	-5904756	-47646003780	-80065270602672
3	0	0	0	66	-45729	627574428	3776946955338
4	0	0	0	0	-132	-453960	-95306132778
5	0	0	0	0	0	-5031	1028427996
6	0	0	0	0	0	-18	-771642
7	0	0	0	0	0	0	-7224
8	0	0	0	0	0	0	-24

Table 2: Some BPS invariants for $n^g_{(d_E,2)}$

❷ Checks form algebraic geometry:

Using the definition of BPS states as Hodge numbers of the BPS moduli space, we get vanishing conditions, from the Castelnouvo bounds, as well as explicite results for non singular moduli spaces:

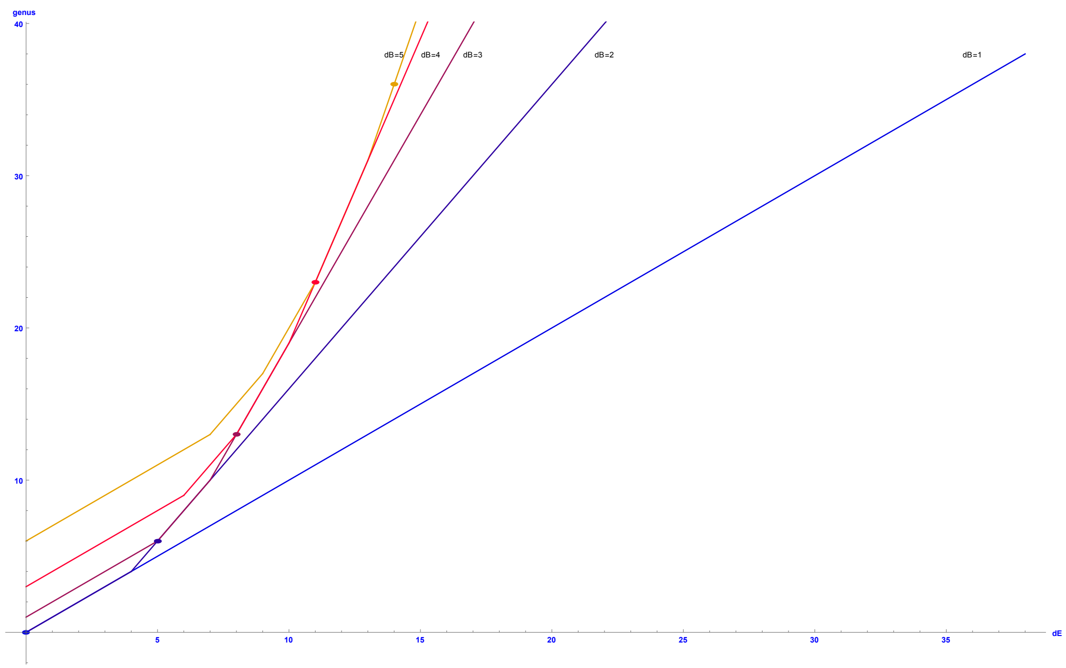


Figure 1: The figure shows the boundary of non-vanishing curves for the values of $d_B = 1, 2, 3, 4, 5$.

Computing the Euler characteristic of the BPS moduli space, we obtain for these values on the edges of the figure

$$n_{d_E,d_B}^{d_E d_B-(3d_B^2-d_B-2)/2} = (-1)^{d_E d_B-(1/2)(3d_B^2+d_B-4)} \, 3\left(d_E d_B - \frac{3d_B^2+d_B-6}{2}\right) \, .$$

which perfectly matches the predction of the weak Jacobi forms.

☆ Conclusions:

$$Z_{d_B}(\tau, z) = \frac{\varphi_{d_B}(\tau, z)}{\eta^{36d_B}(\tau) \prod_{k=1}^{d_B} \varphi_{-2,1}(\tau, kz)}. \tag{5}$$

➼ Since the elliptic argument z of the Jacobi forms is identified with the string coupling

$$g_s = 2\pi i z$$

this expression captures all genus contributions for a given base class.

➼ From the transformation properties of weak Jacobi-forms it follows that the depence of Z on string the coupling is coupling is quasi periodic.

- Since (1) has poles only at the torsion points of the elliptic argument

$$Z_{d_B}(\tau, z) = Z_{d_B}^{pol} + Z_{d_B}^{fin},$$

where the finite part

$$Z_{d_B}^{fin}(\tau, z) = \sum_{l \in \mathbb{Z}/2m\mathbb{Z}} h_l(\tau)\theta_{m,l}(\tau, z)$$

has an expansion in terms of mock modular forms $h_l(\tau)$.

- The latter fact can be used to check the microscopic entropy of 5d $N = 2$ spinning black holes and the wall crossing behaviour of 4d BPS states. Some partial results have been obtained by Vafa et. al. arXiv:1509.00455

- We can make infinitly many checks from algebraic geometry for those curves which have smooth moduli spaces, as seen above. But e.g. for $d_B = 1$ one can confirm the formulas for all classes Jim Bryan et. all work in progress