Integrality of DT invariants and Yangians

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Ingredients of ncDT theory I: representations of quivers

Fix Q a symmetric quiver, vertices Q_0, arrows Q_1, $s, t : Q_1 \to Q_0$ the map taking an arrow to its source or target respectively. Fix $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$.

Definition of refined ncDT invariants

ncDT theory is the study of

$$\underbrace{\mathrm{pLOG}}_{\text{slide after next}} \left(\left[\bigoplus_{\gamma \in \mathbb{N}^{Q_0}} \mathrm{H}(\underbrace{\mathrm{Rep}_\gamma(\mathbb{C}Q)}_{\text{this slide}}, \underbrace{\phi_{\mathrm{Tr}(W)}}_{\text{next slide}} \right] \right) \in \underbrace{\mathrm{K}(\mathrm{MHS}_{\mathbb{Z}^{Q_0}})}_{\text{slide after next}}$$

- Here $\gamma \in \mathbb{N}^{Q_0}$ is the *dimension vector.*
- $\mathrm{Rep}_\gamma(\mathbb{C}Q) = \left(\prod_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\gamma(s(a))}, \mathbb{C}^{\gamma(t(a))}) \right) / \prod_{i \in Q_0} \mathrm{GL}_{\mathbb{C}}(\gamma(i))$ is the stack of γ-dimensional $\mathbb{C}Q$-modules. It is *smooth*.
- As a special case, we can set $W = 0$. Then $\phi_{\mathrm{Tr}(W)} = \mathbb{Q}$, and the cohomology above is just $\prod_{i \in Q_0} \mathrm{GL}_{\mathbb{C}}(\gamma(i))$-equivariant cohomology of $\sum_{a \in Q_1} \gamma(s(a))\gamma(t(a))$-dimensional affine space.

Ingredients of ncDT theory II: potentials and vanishing cycles

Vanishing cycles complex/functor

Let $f : X \to \mathbb{C}$ be a regular function on a smooth complex variety,

$$\phi_f := \varphi_f \mathbb{Q}_X[\dim X - 1] \in \mathrm{Ob}(\mathrm{MHM}(X))$$

is the mixed Hodge module of vanishing cycles, supported on $\mathrm{crit}(f)$.

The element $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ defines a function $\mathrm{Tr}(W)$ on $\mathrm{Rep}_\gamma(\mathbb{C}Q)$.

$$\mathrm{supp}(\phi_{\mathrm{Tr}(W)}) = \mathrm{crit}(\mathrm{Tr}(W)) = \mathrm{Rep}(\mathrm{Jac}(Q, W)) \subset \mathrm{Rep}(\mathbb{C}Q)$$

So we may rewrite

$$\mathrm{H}(\mathrm{Rep}_\gamma(\mathbb{C}Q), \phi_{\mathrm{Tr}(W)}) = \mathrm{H}(\mathrm{Rep}_\gamma(\mathrm{Jac}(Q, W)), \phi_{\mathrm{Tr}(W)}).$$

Conclusion: $\phi_{\mathrm{Tr}(W)}$ is a special sheaf/MHM on the stack of $\mathrm{Jac}(Q, W)$-modules, giving the 'DT cohomology' of this stack.

Ingredients of ncDT theory II: potentials and vanishing cycles

Definition of the Jacobi algebra

$$\mathrm{Jac}(Q, W) := \mathbb{C}Q/\langle \partial W/\partial a | a \in Q_1 \rangle$$

Example

Take $\mathbb{C}Q_3 = \mathbb{C}\langle x, y, z \rangle$ and $W = xyz - xzy$. Then

$$\begin{aligned} \mathrm{Jac}(Q, W) :=& \mathbb{C}Q/\langle \partial W/\partial a | a \in Q_1 \rangle \\ =& \mathbb{C}Q/\langle yz - zy, zx - xz, xy - yx \rangle \\ \cong& \mathbb{C}[x, y, z] \end{aligned}$$

Cryptic remark

While the category of $\mathbb{C}Q$-representations is 1-dimensional, the category of $\mathrm{Jac}(Q, W)$-representations is 3-dimensional and Calabi-Yau. This explains why physicists care about Jacobi algebras.

Ingredients of ncDT theory III: plethystic logarithm

- Let $\mathcal{C}$ be a $\mathbb{Q}$-linear tensor category, then the Grothendieck ring

$$\mathsf{K}(\mathcal{C}) = \mathbb{Z}[[M]_{\mathsf{K}}, M \in \mathrm{Ob}(\mathcal{C}]/\left(\substack{([M']_{\mathsf{K}}+[M'']_{\mathsf{K}})\sim[M]_{\mathsf{K}} \\ \text{if } \exists \text{ s.e.s } M'\to M\to M''}\right)$$

  becomes a $\mathbb{Z}$-algebra, with multiplication $[M]_{\mathsf{K}} \cdot [N]_{\mathsf{K}} := [M \otimes N]_{\mathsf{K}}$. The product on $\mathsf{K}(\mathcal{C})$ is a *decategorification* of $\otimes$.
- Now assume that $\mathcal{C}$ is a symmetric tensor category. What is the decategorification of this *extra* structure?

Ingredients of ncDT theory III: plethystic logarithm

Decategorification of symmetric structure is given by plethystic exponential

$$\mathrm{pEXP} : \mathsf{K}(\mathcal{C}_{\mathbb{Z}_{>0}}) \to \mathsf{K}(\mathcal{C}_{\mathbb{Z}_{\geqslant 0}})$$

sending

$$[M]_{\mathsf{K}} \mapsto [\mathsf{Sym}(M)]_{\mathsf{K}} := [\bigoplus_{i \geqslant 0} \mathsf{Sym}^i(M)]_{\mathsf{K}}.$$

Example

If $\mathbb{Q}_i^n$ is considered as vector space in degree i,
$\mathrm{pEXP}([\mathbb{Q}^n]_{\mathsf{K}}) = [\mathbb{Q}[x_1, \ldots, x_n]]_{\mathsf{K}}$ with generators in degree i.

The map pEXP has an inverse defined on its image:
$\mathrm{pLOG} : 1 + \mathsf{K}(\mathcal{C}_{\mathbb{Z}_{>0}}) \to \mathsf{K}(\mathcal{C}_{\mathbb{Z}_{\geqslant 0}})$.

Example

For example if the generators dx_s are put in degree i,
$\mathrm{pLOG}([\mathbb{Q}[dx_1, \ldots, dx_n]]_{\mathsf{K}}) = [\mathbb{Q}_i^n[1]]_{\mathsf{K}} = -[\mathbb{Q}_i^n]_{\mathsf{K}}.$

The integrality conjecture: heuristic view

Recap

Recap: ncDT theory is the study of the class

$$\mathrm{pLOG}\Big(\Big[\bigoplus_{\gamma\in\mathbb{N}^{Q_0}} \mathsf{H}(\mathsf{Rep}_\gamma(\mathsf{Jac}(Q,W)),\phi_{\mathsf{Tr}(W)}\Big]\Big)\in \mathsf{K}(\mathsf{MHS}_{\mathbb{Z}^{Q_0}})$$

<u>The unattainable ideal</u>: *Imagine* that every object in $\mathsf{Ob}(\mathsf{Jac}(Q,W))$ had a unique, canonical decomposition as a sum of simple objects, and $\mathcal{M}=\coprod_{\gamma\in\mathbb{Z}^{Q_0}}\mathcal{M}_\gamma$ is the variety of simple objects. Then the *stack* of simple objects $\mathcal{M}/\mathbb{C}^*$ has cohomology $\mathsf{H}(\mathcal{M})\otimes\mathsf{H}(\mathrm{pt}/\mathbb{C}^*)$, and the stack of all objects satisfies

$$\begin{aligned}\Big[\mathsf{H}(\mathsf{Rep}(\mathsf{Jac}(Q,W)),\phi_{\mathsf{Tr}(W)})\Big]_{\mathsf{K}} &= \sum_{i\geq 0}\Big[\mathsf{H}((\mathcal{M}/\mathbb{C}^*)^{\times i}/\mathsf{Sym}_i,\phi_{\mathsf{Tr}(W)})\Big]_{\mathsf{K}}\\ &=\sum_{i\geq 0}\Big[\mathsf{H}\left(\mathcal{M}/\mathbb{C}^*,\phi_{\mathsf{Tr}(W)}\right)^{i}/\mathsf{Sym}_i\Big]_{\mathsf{K}}\\ &=\mathrm{pEXP}\Big(\big[\mathsf{H}(\mathcal{M},\phi_{\mathsf{Tr}(W)})[u]\big]_{\mathsf{K}}\Big)\end{aligned}$$

The integrality conjecture

The integrality conjecture

Each $\mathbb{N}^{Q_0}$-graded piece of

$$\mathrm{pLOG}\left(\Big[\bigoplus_{\gamma\in\mathbb{N}^{Q_0}}\mathsf{H}\left(\mathrm{Rep}_\gamma(\mathrm{Jac}(Q,W)),\phi_{\mathrm{Tr}(W)}\right)\Big]_{\mathsf{K}}\right)\in \mathsf{K}(\mathrm{MHS}_{\mathbb{Z}^{Q_0}})$$

is given by $V_\gamma[u]$ for V_γ a finite dimensional mixed Hodge structure.

I.e. the conjecture says that although the dream situation of the previous slide is *nonsense*, there is still a mixed Hodge structure V_γ that functions as the mythical $\mathsf{H}(\mathcal{M}_\gamma,\phi_{\mathrm{Tr}(W)})$.

The analogy with Yangians

Definition of Yangians

Let $\mathfrak{g}$ be a semisimple Lie algebra associated to Dynkin diagram Q. The $\mathbb{N}^{Q_0}$-graded Lie algebra $\mathfrak{g}[u]$ is defined by the rule $[gu^a, g'u^b] = [g, g']u^{a+b}$. The Yangian $Y(\mathfrak{g})$ is a deformation of $U(\mathfrak{g}[u])$. I.e. there is a filtration F of $Y(\mathfrak{g})$ such that

$$\mathrm{Gr}_F(Y(\mathfrak{g})) \cong U(\mathfrak{g}[u])$$

as algebras.

The PBW type theorem for these quantum groups states that the map

$$\mathrm{Sym}(\mathfrak{g}[u]) \to Y(\mathfrak{g})$$

is an isomorphism of $\mathbb{N}^{Q_0}$-graded vector spaces. Put more suggestively:

$$\mathrm{pLOG}([Y(\mathfrak{g})]_{\mathsf{K}}) = [\mathfrak{g}[u]]_{\mathsf{K}} = [\mathfrak{g} \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*)]_{\mathsf{K}}.$$

The Cohomological Hall algebra

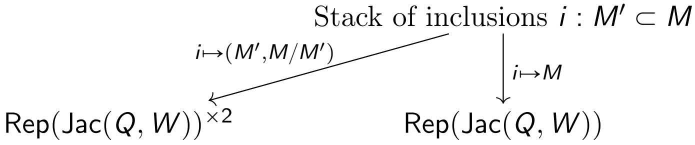
- In terms of plethystic exponentiation:
  $\mathsf{K}\,(\text{PBW isom for } Y(\mathfrak{n}_+)) = \text{Integrality conjecture for } [Y(\mathfrak{n}_+)]_{\mathsf{K}}$.
- The idea, then, is to *realise* $\mathcal{H}_{Q,W} := \bigoplus_{\gamma\in\mathbb{N}^{Q_0}} \mathrm{H}(\mathrm{Rep}_\gamma(\mathbb{C}Q), \phi_{\mathrm{Tr}(W)})$ as a *generalised* (half of a) Yangian, proving the integrality conjecture.
- Via the correspondence diagram

$$\begin{array}{ccc} & & \text{Stack of inclusions } i : M' \subset M \\ & \swarrow^{\,i\mapsto(M',M/M')} & \downarrow{\scriptstyle i\mapsto M} \\ \mathrm{Rep}(\mathrm{Jac}(Q,W))^{\times 2} & & \mathrm{Rep}(\mathrm{Jac}(Q,W)) \end{array}$$

  Kontsevich & Soibelman build Hall algebra structure on $\mathcal{H}_{Q,W}$ by pulling back DT-cohomology along the diagonal map, and pushing forward along the vertical one.
- So $\mathcal{H}_{Q,W}$ is at least an algebra! But what about all the extra structure that a Yangian is supposed to have?

The coproduct

- Something we haven't mentioned yet is the coproduct on $Y(\mathfrak{g}[u])$.
- In fact (see e.g. Maulik & Okounkov's very long paper) there is an expectation that in defining a coproduct $\Delta : Y(\mathfrak{g}) \to Y(\mathfrak{g}) \otimes Y(\mathfrak{g})$ we will have to localize the target.
- There is a very natural candidate for the coproduct coming from the *same* correspondence diagram



- Namely, pull pack cohomology along the vertical map then push forward along the diagonal one.
- The expected localizations appear naturally in the definition of this pushforward.

The relative CoHA

The tensor category MHS

The category MHM(pt) is the same as the category MHS of mixed Hodge structures. The scheme pt is a symmetric monoid via the map $\tau : \mathrm{pt} \times \mathrm{pt} \to \mathrm{pt}$. We recover the usual tensor structure on MHS by the definition $\mathcal{F}' \boxtimes_{\tau} \mathcal{F}'' := \tau_* (\mathcal{F}' \boxtimes \mathcal{F}'')$ for $\mathcal{F}', \mathcal{F}' \in \mathrm{MHM}(\mathrm{pt})$.

In the same way, whenever we have a symmetric monoid $(X, \tau : X \times X \to X, \mathrm{pt} \to X)$ we obtain a symmetric tensor structure on $\mathrm{MHM}(X)$

Example

We've already seen an example in action. A $\mathbb{N}^{Q_0}$*-graded mixed Hodge structure* is the same as a mixed Hodge module on $\mathbb{N}^{Q_0}$. A $\mathbb{N}^{Q_0}$*-graded algebra in* MHS is the same as an algebra in the tensor category $\mathrm{MHM}(\mathbb{N}^{Q_0})$ with tensor product $\boxtimes_+$, where $+ : \mathbb{N}^{Q_0} \times \mathbb{N}^{Q_0} \to \mathbb{N}^{Q_0}$ is the addition map.

The relative CoHA

Definition

Given a map of scheme monoids $f : (X, \tau) \to (Y, \tau')$, and an algebra $(A, m_A : A \boxtimes_{\tau'} A \to A)$ in $\mathrm{MHM}(Y)$, we say an algebra $(B, m_B : B \boxtimes_{\tau} B \to B)$ is a *lift* of A if $f_* B \cong A$ and this isomorphism commutes with the multiplication.

Example

Let X_Q be the variety parameterizing direct sums of simple $\mathbb{C}Q$-modules. X_Q is a monoid via $(M, M') \mapsto M \oplus M'$. The map $X_Q \xrightarrow{M \mapsto \dim(M)} \mathbb{N}^{Q_0}$ is a monoid homomorphism.

Key observation

The CoHA

$$\mathcal{H}_{Q,W} := (\mathrm{Rep}(\mathrm{Jac}(Q, W)) \to \mathbb{N}^{Q_0})_* \phi_{\mathrm{Tr}(W)},$$

an algebra in $\mathrm{MHM}(\mathbb{N}^{Q_0})$, possesses a lift to $\mathrm{MHM}(X_Q)$:

$$\overline{\mathcal{H}}_{Q,W} := (\mathrm{Rep}(\mathrm{Jac}(Q, W)) \xrightarrow{Jordan-Holder} X_Q)_* \phi_{\mathrm{Tr}(W)}$$

Cohomological approximation

Why is it easier to work with $\overline{\mathcal{H}}_{Q,W} := (\mathrm{Rep}(\mathbb{C}Q) \xrightarrow{\mathtt{JH}} X_Q)_* \phi_{\mathrm{Tr}(W)}$?

Framed representations

The map JH is approximated by proper maps:

- Let $\mathrm{frRep}_{\gamma,n}(\mathbb{C}Q)$ be the space of pairs (M, f), where M is a γ-dimensional $\mathbb{C}Q$-module and $f : \mathbb{C}^n \to M$ is a morphism of vector spaces, such that $\mathrm{Im}(f)$ generates M. $\mathrm{frRep}_{\gamma,n}(\mathbb{C}Q)$ is *smooth*.
- Then for fixed cohomological degree i, and fixed $\gamma \in \mathbb{N}^{Q_0}$, and $n \gg 0$

$$\mathcal{H}^i(\mathtt{JH}_* \phi_{\mathrm{Tr}(W)})_\gamma = \mathcal{H}^i\left((\mathrm{frRep}_{\gamma,n}(\mathbb{C}Q) \to X_{Q,\gamma})_* \phi_{\mathrm{Tr}(W)}\right)$$

Proof of the integrality conjecture

Since we can 'approximate' $\mathtt{JH}: \mathrm{Rep}(\mathbb{C}Q) \to X_Q$ by proper maps we get to treat $\mathtt{JH}$ *as* a proper map of schemes. This helps!:

1. The vanishing cycles functor commutes with proper maps. So $\overline{\mathcal{H}}_{Q,W} := \left(\mathrm{Rep}(\mathbb{C}Q) \xrightarrow{\mathtt{JH}} X_Q\right)_* \phi_{\mathrm{Tr}(W)} \cong \varphi_{\mathrm{Tr}(W)} \mathtt{JH}_* \mathbb{Q}_{\mathrm{Rep}(\mathbb{C}Q)}$
2. Proper maps send pure MHMs to pure MHMs. So $\mathtt{JH}_* \mathbb{Q}_{\mathrm{Rep}(\mathbb{C}Q)} \in D^b(\mathrm{MHM}(X))$ is pure.
3. A pure MHM $\mathcal{H} \in D^b(\mathrm{MHM}(X))$ is determined by $[\mathcal{H}]_{\mathrm{K}} \in \mathrm{K}(\mathrm{MHM}(X))$.

Theorem (Meinhardt-Reineke)

$$[\mathtt{JH}_* \mathbb{Q}_{\mathrm{Rep}(\mathbb{C}Q)}]_{\mathrm{K}} = \mathrm{pEXP}([\mathrm{IC}' \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*)]_{\mathrm{K}}) \in \mathrm{K}(\mathrm{MHM}(X_Q))$$

IC' is a small modification of the intersection cohomology sheaf of X_Q. $\mathrm{IC}'_\gamma = 0$ if there are no stable γ-dimensional Q-reps, it is $\mathrm{IC}(\mathbb{Q}_{X^{st}_{Q,\gamma}}[\dim(X^{st}_{Q,\gamma}])$ otherwise.

This is enough to prove the integrality conjecture.

The integrality theorem

Theorem: Doubly refined integrality conjecture

$\exists$ isomorphism $\overline{\mathcal{H}}_{Q,W} := \mathtt{JH}_* \phi_{\mathrm{Tr}(W)} \cong \mathrm{Sym}_{\boxtimes_\oplus} \left(\varphi_{\mathrm{Tr}(W)} \mathrm{IC}' \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*) \right)$.

$\mathcal{H}_{Q,W} = \dim_* \mathtt{JH}_* \phi_{\mathrm{Tr}(W)} \cong \mathrm{Sym} \left(\dim_* (\varphi_{\mathrm{Tr}(W)} \mathrm{IC}' \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*)) \right)$

1. The first refinement is that we work at the level of $\mathrm{MHM}(X_Q)$, not $\mathrm{MHM}(\mathbb{N}^{Q_0})$.
2. The second refinement is that we don't have to pass to $\mathrm{K}(\mathrm{MHM}(X_Q))$ – this is a categorified version of the integrality conjecture.

The perverse filtration

The fact that we can 'approximate' $\mathtt{JH}: \mathrm{Rep}(\mathrm{Jac}(Q,W)) \to X_Q$ by proper maps provides another piece of the link with Yangians. The cohomology of the RHS of

$$\mathrm{H}^i(\mathrm{Rep}_\gamma(\mathbb{C}Q), \phi_{\mathrm{Tr}(W)}) \cong \mathbb{H}^i\left((\mathrm{frRep}_{\gamma,n} \to X_{Q,\gamma})_* \phi_{\mathrm{Tr}(W)}\right) \quad n \gg 0$$

acquires a perverse filtration from the perverse t-structure on X_Q. So $\mathcal{H}_{Q,W}$ acquires a perverse filtration F too.

1. The product and coproduct on the CoHA $\mathcal{H}_{Q,W}$ preserve the filtration F.
2. Passing to the associated graded $\mathrm{Gr}_F(\mathcal{H}_{Q,W})$, the elements $\dim_* \varphi_{\mathrm{Tr}(W)} \mathrm{IC}' \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*)$ are primitive (i.e. $\Delta(v) = 1 \otimes v + v \otimes 1$)
3. So $U(\dim_* \varphi_{\mathrm{Tr}(W)} \mathrm{IC}' \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*)) \to \mathcal{H}_{Q,W}$ is an injection, and so an isomorphism!

The CoHA is a Yangian!

Theorem

For arbitrary (Q, W)*, the algebra* $\mathcal{H}_{Q,W}$ *is a quantum enveloping algebra for some* $\mathbb{N}^{Q_0}$*-graded Lie algebra* $\mathfrak{g}_{Q,W}[u]$*. So the DT invariants for* $\mathrm{Jac}(Q, W)$ *are given by* $[\mathfrak{g}_{Q,W,\gamma}] \in \mathrm{K}(\mathrm{MHM})$ *for* $\gamma \in \mathbb{N}^{Q_0}$*.* $\mathfrak{g}_{Q,W,\gamma}$ *is given explicitly by* $\mathrm{H}(X_\gamma, \varphi_{\mathrm{Tr}(W)} \mathrm{IC}'_\gamma)$*, so in particular the integrality conjecture holds.*

Example

For any quiver Q there is a quiver $\tilde{Q}$ with potential W such that $\mathfrak{g}_{\tilde{Q},W}$ has as (cohomological) degree zero piece the Kac-Moody Lie algebra associated to Q.

Example

For genus $g \geqslant 1$ there is a smooth algebra B_g with potential W_g such that $\mathfrak{g}_{B_g,W_g,n}$ is isomorphic to the cohomology of the (twisted) character variety of n-dimensional representations of the fundamental group of the genus g Riemann surface.