Bihamiltonian cohomology and deformations of Poisson pencils

Guido Carlet

Korteweg - de Vries Institute for Mathematics, Amsterdam University

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Outline

Outline

KdV

The Korteweg - de Vries equation

$$u_t = uu_x + \epsilon^2 u_{xxx}$$

has bihamiltonian formulation

$$u_t(x) = \{u(x), H_1\}_1 = \{u(x), H_0\}_2$$

with compatible Poisson brackets

$$\{u(x), u(y)\}_1 = \delta'(x-y),$$
$$\{u(x), u(y)\}_2 = u(x)\delta'(x-y) + \frac{1}{2}u'(x)\delta(x-y) + \frac{3}{2}\epsilon^2\delta'''(x-y).$$

(Gardner-Zakharov-Faddeev'71, Magri'78)

General problem

Scalar case $N = 1$

Classify (bi)hamiltonian structures of the form

$$\{u(x), u(y)\} = \{u(x), u(y)\}^0 + \\ + \sum_{m \geqslant 2} \sum_{l=0}^{m+1} \epsilon^m A_{m,l}(u; u_x, \dots) \delta^{(l)}(x - y)$$

under Miura type transformations

$$u(x) \to u(x) + \epsilon f_1(u; u_x) + \epsilon^2 f_2(u; u_x, u_{xx}) + \dots$$

where $A_{m,l}$, f_i are differential polynomials.

(Dubrovin-Zhang'01)

Outline

Poisson brackets of Dubrovin-Novikov type

Leading order:

$$\{u^i(x), u^j(y)\}^0 = g^{ij}(u(x))\delta'(x-y) + \Gamma^{ij}_k(u(x))u^k_x(x)\delta(x-y),$$

is a Poisson structure iff

- g^{ij} flat contravariant metric,
- Γ^{ij}_k Christhoffel symbols of g^{ij}.

(Dubrovin-Novikov'83)

Semisimple Poisson pencil of DN type

- Compatible Poisson brackets of DN type

$$\{w^i(x), w^j(y)\}_1^0, \quad \{w^i(x), w^j(y)\}_2^0$$

i.e., g_1^{ij}, g_2^{ij} flat pencil of metrics.

- Semisimple when $\det(g_2(w) - \lambda g_1(w)) = 0$ has pairwise distinct real roots in $\lambda = u^1(w), \ldots, u^n(w)$.

- $u^1, \ldots, u^n$ are canonical coordinates, i.e., the metrics are diagonal:

$$g_1^{ij} = f^i(u)\delta_{ij}, \quad g_2^{ij} = u^i f^i(u)\delta_{ij}.$$

Local multivectors

- In finite dimensions: the space Λ^* of multivectors on a manifold M is endowed with the Schouten-Nijenhuis bracket

$$[,]:\Lambda^p\times\Lambda^q\to\Lambda^{p+q-1}$$

- On a formal loop space $\mathcal{L}M=\{S^1\to M\}$: one considers the space Λ^*_{loc} of local multivectors of the form (for $M=\mathbb{R}$)

$$\sum_{p_2\cdots p_k\geqslant 0} B_{p_2\dots p_k}(u(x);u_x(x),u_{xx}(x),\dots)\delta^{(p_2)}(x-x_2)\cdots\delta^{(p_k)}(x-x_k)$$

which is closed under a suitably defined Schouten-Nijenhuis bracket

$$[,]:\Lambda^p_{loc}\times\Lambda^q_{loc}\to\Lambda^{p+q-1}_{loc}$$

Poisson cohomology and deformations

- A bivector $P \in \Lambda^2_{loc}$ is a Poisson structure iff $[P, P] = 0$
$\Longrightarrow d_P := [P, \cdot] : \Lambda_{loc} \to \Lambda_{loc}$ is a differential $d_P^2 = 0$.

- Let $P \in \Lambda^2_{loc}$ Poisson bivector. The Poisson cohomology of P is
$$H(\Lambda_{loc}, d_P) = \frac{\mathrm{Ker}\, d_P}{\mathrm{Im}\, d_P}.$$

- The Poisson cohomology $H(\Lambda_{loc}, d_P)$ of a Poisson structure of DN type P vanishes in positive degree. (Getzler'02)

- All deformations of a single Poisson structure of DN type are trivial under Miura transformations.

Deformations of bihamiltonian structure

- Deformation theory of a Poisson pencil P_1, P_2 of hydrodynamic type is governed by bihamiltonian cohomology groups

$$BH(\Lambda_{loc}, d_1, d_2) = \frac{\operatorname{Ker} d_1 \cap \operatorname{Ker} d_2}{\operatorname{Im} d_1 d_2}$$

where $d_i = [P_i, \cdot]$.

- Infinitesimal deformations ($O(\epsilon^3)$) are classified by $BH^2(\Lambda_{loc})$, i.e., by central invariants

$$c_i(u) = \frac{1}{3(f^i(u))^2}\left(A^{ii}_{2,3;2} - u^i A^{ii}_{2,3;1} + \sum_{k\neq i} \frac{(A^{ij}_{1,2;2} - u^i A^{ij}_{1,2;1})^2}{f^k(u)(u^k - u^i)}\right).$$

(Liu-Zhang'05, Dubrovin-Liu-Zhang'06)

Existence of deformations

- Given an infinitesimal deformation of a Poisson pencil of DN type, is it possible to extend it to a full dispersive Poisson pencil ?

  Main Theorem (C-Posthuma-Shadrin'15):
  The deformations of any semisimple Poisson pencil of DN type are unobstructed.

- Previously known for the dKdV Poisson pencil. (Liu-Zhang'13)
- Sufficient to show that $BH^3_{\geqslant 5}(\Lambda_{loc}, d_1, d_2)$ vanishes.

Outline

Our results:

1 We compute the full bihamiltonian cohomology of the dispersionless KdV Poisson pencil:

Theorem (C-Posthuma-Shadrin'14)

The bihamiltonian cohomology of the dispersionless KdV Poisson pencil is given by

$$BH^p_d(\Lambda_{loc}, d_1, d_2) \cong \begin{cases} C^\infty(\mathbb{R}) & \text{for } (p,d) = (1,1),(2,1),(2,3),(3,3) \\ \mathbb{R} & \text{for } (p,d) = (0,0) \\ 0 & \text{otherwise.} \end{cases}$$

2 We generalize the above result, computing the full bihamiltonian cohomology of general scalar Poisson pencil of hydrodynamic type. (C-Posthuma-Shadrin'15-a)

Our results:

3 We show that the bihamiltonian cohomology of a semisimple Poisson pencil of hydrodynamic type with n dependent variables vanishes but for a finite number of bi-degrees:

Theorem (C-Posthuma-Shadrin'15-b)

The bihamiltonian cohomology $BH^p_d(\Lambda_{loc}, d_1, d_2)$ vanishes for all bi-degrees (p, d) with $d \geqslant 2$, unless

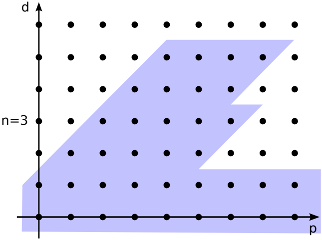
$$d = 2, \dots, n, \quad p = d, \dots, d + n,$$

$$d = n + 1, n + 2, \quad p = d, \dots, d + n - 1.$$

For example, in the $n = 3$ case, we claim the bihamiltonian cohomology

$$BH^p_d(\Lambda_{loc}, d_1, d_2)$$

vanishes in all bi-degrees but those highlighted.



In particular, this implies the vanishing of $BH^3_{\geqslant 5}(\Lambda_{loc})$ which in turn implies the vanishing of the obstructions.

Outline

Supervariables formalism

Consider the space

$$\hat{\mathcal{A}} := C^\infty(\mathbb{R})[[u^1, u^2, \dots; \theta, \theta^1, \dots]]$$

of formal series

$$f(u; u^1, u^2, \dots; \theta, \theta^1, \dots) \in \hat{\mathcal{A}}$$

in the commuting variables $u^1, u^2, \dots$ and in the anticommuting variables $\theta, \theta^1, \theta^2, \dots$.

- x-derivative: $\partial = \sum_{s \geqslant 0} \left(u^{s+1} \frac{\partial}{\partial u^s} + \theta^{s+1} \frac{\partial}{\partial \theta^s} \right) : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$
- two gradations:

$$\hat{\mathcal{A}}^p_d = \text{homogeneous component with degree } \begin{cases} p \text{ in } \theta, \theta^1, \dots \\ d \text{ in } x\text{-derivatives.} \end{cases}$$

- Let $\hat{\mathcal{F}} := \frac{\hat{\mathcal{A}}}{\partial \hat{\mathcal{A}}}$ and denote the projection map $\int : \hat{\mathcal{A}} \to \hat{\mathcal{F}}$.
- $\Lambda^p_{loc} \cong \hat{\mathcal{F}}^p$
- The Schouten-Nijenhuis bracket is

$$[,] : \hat{\mathcal{F}}^p \times \hat{\mathcal{F}}^q \to \hat{\mathcal{F}}^{p+q-1}$$

$$[P, Q] = \int \left(\delta^\bullet P \delta_\bullet Q + (-1)^p \delta_\bullet P \delta^\bullet Q \right)$$

$$\delta^\bullet = \sum_{s \geqslant 0} (-\partial)^s \frac{\partial}{\partial \theta^s}, \quad \delta_\bullet = \sum_{s \geqslant 0} (-\partial)^s \frac{\partial}{\partial u^s}$$

- A bivector $P \in \hat{\mathcal{F}}^2$ is a Poisson structure iff $[P, P] = 0$.
- By (graded) Jacobi identity $d_P := [P, \cdot] : \hat{\mathcal{F}} \to \hat{\mathcal{F}}$ is a differential $d_P^2 = 0$.

- It is more convenient to work in $\hat{\mathcal{A}}$ rather than in $\hat{\mathcal{F}}$.

(Liu-Zhang'13)

- For any $P \in \hat{\mathcal{F}}^2$, let $d_P = [P, \cdot]$, there exists a map D_P s.t. the diagram commutes

$$\begin{array}{ccc} \hat{\mathcal{A}} & \xrightarrow{D_P} & \hat{\mathcal{A}} \\ \Big\downarrow \int & & \Big\downarrow \int \\ \hat{\mathcal{F}} & \xrightarrow{d_P} & \hat{\mathcal{F}} \end{array}$$

which is given by

$$D_P = \sum_{s \geqslant 0} \left(\partial^s (\delta^\bullet P) \frac{\partial}{\partial u^s} + \partial^s (\delta_\bullet P) \frac{\partial}{\partial \theta^s} \right)$$

- The short exact sequence of complexes above gives rise to a long exact sequence in cohomology that allow to recover the cohomology of $\hat{\mathcal{F}}$ from the cohomology of $\hat{\mathcal{A}}$.

Polynomial complexes and Barakat lemma

Let us consider the related polynomial complex

$$(\hat{\mathcal{F}}[\lambda], d_\lambda), \qquad d_\lambda = d_2 - \lambda d_1.$$

For almost all (p, d) the bihamiltonian cohomology groups are isomorphic to the cohomology groups of the corresponding polynomial complex i.e.

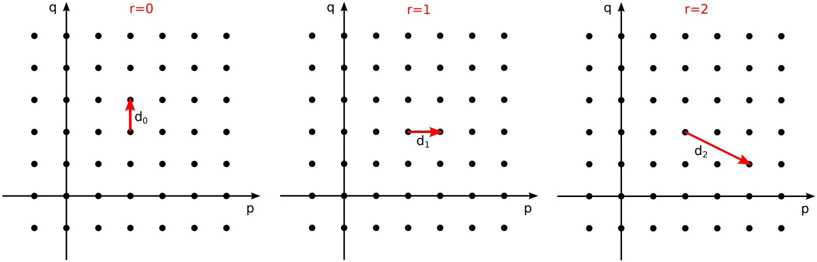
$$BH^p_d(\hat{\mathcal{F}}, d_1, d_2) \cong H^p_d(\hat{\mathcal{F}}[\lambda], d_\lambda)$$

for $p, d \geqslant 0$ excluding $(p, d) = (0, 0), (1, 0), (1, 1), (2, 1)$.

(Barakat'08, Liu-Zhang'13)

Filtrations and spectral sequences

A (cohomological type) spectral sequence is a family of differential $\mathbb{Z}$-bigraded vector spaces $(E_r^{*,*}, d_r)$ with differentials d_r of bidegree $(r, 1-r)$



such that for all $p, q \in \mathbb{Z}$ and all $r \geqslant 0$

$$E_{r+1}^{pq} \cong H^{pq}(E_r^{*,*}, d_r) := \frac{\mathrm{Ker}(d_r : E_r^{pq} \to E_r^{p+r,q-r+1})}{\mathrm{Im}(d_r : E_r^{p-r,q+r-1} \to E_r^{pq})}.$$

(C, d) - filtered $\mathbb{Z}$-graded differential complex

- $F^i C$, $i \in \mathbb{Z}$ - decreasing filtration of (C, d)

$$\cdots \subset F^{i+1} \subset F^i C \subset \cdots \subset C$$

- $d(F^i C) \subset F^i C$ - filtration is preserved by differential

With a filtered Z-graded differential complex one associates a spectral sequence $(E_r^{*,*}, d_r)$ with

$$E_0^{p,q} = gr^p C^{p+q}$$

$$E_1^{p,q} = \frac{d^{-1}(F^{p+1} C^{p+q+1}) \cap F^p C^{p+q}}{d(F^p C^{p+q-1}) + F^{p+1} C^{p+q}},$$

with differentials d_0, d_1 induced by d on the quotients.

The cohomology of a filtered graded complex (C, d) inherits a filtration, where $F^i H(C, d)$ is given by the image of $H(F^i C, d)$ in $H(C, d)$ under the inclusion map.

Theorem

The spectral sequence associated with a bounded filtration converges to $H(C, d)$, i.e.,

$$E_\infty^{p,q} \cong \frac{F^p H^{p+q}(C, d)}{F^{p+1} H^{p+q}(C, d)}$$

A filtration $F^* C$ is bounded if for each degree p there are integers s and t such that

$$0 = F^s C^p \subset \cdots \subset F^{i+1} C^p \subset F^i C^p \subset \cdots \subset F^t C^p = C^p.$$

Outline

KdV case

- The dispersionless KdV Poisson bivectors are represented by the elements in $\hat{\mathcal{F}}$

$$P_1 = \frac{1}{2}\int \theta\theta^1, \qquad P_2 = \frac{1}{2}\int u\theta\theta^1.$$

- The differentials on $\hat{\mathcal{F}}$ induced by the Schouten bracket are

$$d_i = d_{P_i} = [P_i, \cdot], \quad i = 1, 2.$$

- The corresponding differentials on $\hat{\mathcal{A}}$ are

$$D_1 = \sum_{s \geqslant 0} \theta^{s+1} \frac{\partial}{\partial u^s},$$
$$D_2 = \sum_{s \geqslant 0} \left(\partial^s (u\theta^1 + \frac{1}{2} u_1 \theta) \frac{\partial}{\partial u^s} + \partial^s (\frac{1}{2}\theta\theta^1) \frac{\partial}{\partial \theta^s} \right).$$

Main problem

Our main problem is to compute the cohomology of the complex

$$(\hat{\mathcal{A}}[\lambda], D_\lambda)$$

where

$$\hat{\mathcal{A}} = C^\infty(\mathbb{R})[[u^1, u^2, \ldots; \theta, \theta^1, \ldots]]$$

and

$$D_\lambda = \sum_{s \geqslant 0} \left[\partial^s((u-\lambda)\theta^1 + \frac{1}{2} u^1 \theta) \frac{\partial}{\partial u^s} + \partial^s(\frac{1}{2}\theta\theta^1) \frac{\partial}{\partial \theta^s} \right].$$

A filtration of $\hat{\mathcal{A}}[\lambda]$

We define a filtration of $\hat{\mathcal{A}}[\lambda]$

$$F^i \hat{\mathcal{A}}_d[\lambda] = \hat{\mathcal{A}}_d^{(d-i)}[\lambda]$$

by imposing the upper bound $d-i$ on the highest derivative appearing in homogeneous component of standard degree d.

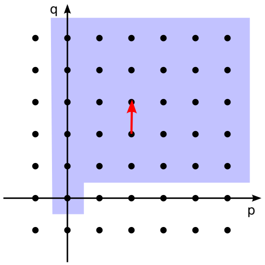
This filtration is bounded:

$$0 = F^{d+1}\hat{\mathcal{A}}_d[\lambda] \subset \cdots \subset F^{i+1}\hat{\mathcal{A}}_d[\lambda] \subset F^i\hat{\mathcal{A}}_d[\lambda] \subset \cdots \subset F^0\hat{\mathcal{A}}_d[\lambda] = \hat{\mathcal{A}}_d[\lambda].$$

We associate with this filtration a spectral sequence $E_r^{p,q}$.

Lemma: The zeroth page $E_0^{*,*}$ of the spectral sequence

$$E_0^{pq} = gr^p \hat{\mathcal{A}}_{p+q}[\lambda] \cong \hat{\mathcal{A}}_{p+q}^{[q]}[\lambda]$$

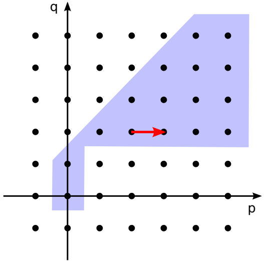


$$d_0 : E_0^{p,q} \to E_0^{p,q+1}$$

$$d_0 = \left((u-\lambda)\theta^{q+1} + \frac{1}{2} u^{q+1} \theta \right) \frac{\partial}{\partial u^q} + \frac{1}{2} \theta \theta^{q+1} \frac{\partial}{\partial \theta^q}$$

Lemma: The first page $E_1^{*,*}$

$$E_1^{p,q} = \begin{cases} \mathbb{R}[\lambda], & p = q = 0 \\ \frac{C^\infty(\mathbb{R})}{\mathbb{R}[u]}\theta\theta^1, & p = 0, q = 1 \\ \hat{\mathcal{A}}_p^{[q-1]}\theta\theta^q & p \geqslant 1, q \geqslant 2. \end{cases}$$



$$d_1 : E_1^{p,q} \to E_1^{p+1,q}$$

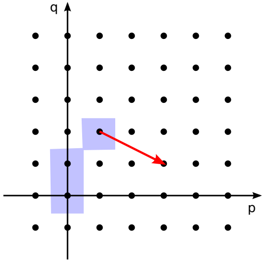
$$d_1(f\theta\theta^q) = \left((D_\lambda(f))_{\lambda=u} + \frac{q-2}{2}\theta^1 f \right) \theta\theta^q$$

Lemma: The second page $E_2^{*,*}$

Important: The following operator is a contracting homotopy of d_1 for $p \geqslant 1$, $q \geqslant 2$ and $(p,q) \neq (1,2)$

$$\left(\sum_{s \geqslant 1} \frac{s+2}{2} u^s \frac{\partial}{\partial u^s} + \sum_{s \geqslant 0} \frac{s-1}{2} \theta^s \frac{\partial}{\partial \theta^s} \right)^{-1} \frac{\partial}{\partial \theta^1}$$

$$E_2^{p,q} = \begin{cases} \mathbb{R}[\lambda] & p=0, q=0, \\ \frac{C^\infty(\mathbb{R})}{\mathbb{R}[u]} \theta\theta^1 & p=0, q=1 \\ C^\infty(\mathbb{R})\theta\theta^1\theta^2 & p=1, q=2 \\ 0 & \text{else.} \end{cases}$$



$$d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$$

The differential d_2 is zero $\to$ the spectral sequence stabilizes

Main proposition

By the convergence theorem for spectral sequences we have

$$E_2^{p,q} = E_\infty^{p,q} \cong \frac{F^p H_{p+q}(\hat{\mathcal{A}}[\lambda], D_\lambda)}{F^{p+1} H_{p+q}(\hat{\mathcal{A}}[\lambda], D_\lambda)}$$

and because the filtration is bounded we have

$$F^0 H_n(\hat{\mathcal{A}}[\lambda], D_\lambda) = H_n(\hat{\mathcal{A}}[\lambda], D_\lambda), \quad F^n H_n(\hat{\mathcal{A}}[\lambda], D_\lambda) = 0.$$

Proposition

The cohomology of the polynomial complex $(\hat{\mathcal{A}}[\lambda], D_\lambda)$ is

$$H(\hat{\mathcal{A}}[\lambda], D_\lambda) = \mathbb{R}[\lambda] \oplus (C^\infty(\mathbb{R})/\mathbb{R}[u])\theta\theta^1 \oplus C^\infty(\mathbb{R})\theta\theta^1\theta^2$$

Main result

By the long exact sequence argument and the Barakat lemma, we derive the bihamiltonian cohomology of $\hat{\mathcal{F}}$ from the cohomology of the complex $(\hat{\mathcal{A}}[\lambda], D_\lambda)$.

Theorem

The bihamiltonian cohomology of the dispersionless KdV Poisson pencil is given by

$$BH^p_d(\hat{\mathcal{F}}, d_1, d_2) \cong \begin{cases} C^\infty(\mathbb{R}) & \text{for } (p,d) = (1,1),(2,1),(2,3),(3,3) \\ \mathbb{R} & \text{for } (p,d) = (0,0) \\ 0 & \text{otherwise.} \end{cases}$$

Remark: This result generalizes to the general scalar case.

Outline

Semisimple n-dimensional case

- Space of local multivectors:

$$\hat{\mathcal{F}} = \frac{\hat{\mathcal{A}}}{\partial \hat{\mathcal{A}}},$$

$$\hat{\mathcal{A}} = C^\infty(U)[[u^{i,1}, u^{i,2}, \dots; \theta_i^0, \theta_i^1, \theta_i^2, \dots]]$$

  with $U \subset \mathbb{R}^n$.

- Poisson brackets $\{,\}_a^0$ as elements in $\hat{\mathcal{F}}^2$:

$$P_a = \frac{1}{2} \int \left(g_a^{ij} \theta_i^0 \theta_j^1 + \Gamma_{k,a}^{ij} u^{k,1} \theta_i \theta_j \right), \quad a = 1, 2.$$

- As before we associate to $P_a \in \hat{\mathcal{F}}^2$ a differential operator D_a on $\hat{\mathcal{A}}$, and define

$$D_\lambda = D_2 - \lambda D_1.$$

- Compute the cohomology

$$H(\hat{\mathcal{A}}[\lambda], D_\lambda).$$

Explicitly

$$D_\lambda = D(u^1 f^1, \ldots, u^n f^n) - \lambda D(f^1, \ldots, f^n)$$

where

$$D(f^1, \ldots, f^n) = \sum_{s \geqslant 0} \partial^s \left(f^i \theta_i^1\right) \frac{\partial}{\partial u^{i,s}}$$

$$+ \frac{1}{2} \sum_{s \geqslant 0} \partial^s \left(\partial_j f^i u^{j,1} \theta_i^0 + f^i \frac{\partial_i f^j}{f^j} u^{j,1} \theta_j^0 - f^j \frac{\partial_j f^i}{f^i} u^{i,1} \theta_j^0 \right) \frac{\partial}{\partial u^{i,s}}$$

$$+ \frac{1}{2} \sum_{s \geqslant 0} \partial^s \left(\partial_i f^j \theta_j^0 \theta_j^1 + f^j \frac{\partial_j f^i}{f^i} \theta_i^0 \theta_j^1 - f^j \frac{\partial_j f^i}{f^i} \theta_j^0 \theta_i^1 \right) \frac{\partial}{\partial \theta_i^s}$$

$$+ \frac{1}{2} \sum_{s \geqslant 0} \partial^s \Bigg(f^j \frac{\partial_i f^l}{f^l} \frac{\partial_i f^l}{f^l} u^{l,1} \theta_l^0 \theta_j^0 - f^l \frac{\partial_i f^l}{f^l} \frac{\partial_l f^j}{f^j} u^{j,1} \theta_l^0 \theta_j^0$$

$$+ f^l \frac{\partial_l f^i}{f^i} \frac{\partial_i f^j}{f^j} u^{j,1} \theta_l^0 \theta_j^0 - \frac{f^l f^j}{f^i} \frac{\partial_l f^i}{f^i} \frac{\partial_j f^i}{f^i} u^{i,1} \theta_l^0 \theta_j^0$$

$$+ f^l \frac{\partial_l f^i}{f^i} \frac{\partial_l f^j}{f^j} u^{j,1} \theta_j^0 \theta_i^0 - f^j \frac{\partial_l f^i}{f^i} \frac{\partial_j f^l}{f^l} u^{l,1} \theta_j^0 \theta_i^0$$

$$+ f^l \frac{\partial_l f^i}{f^i} \frac{\partial_j f^l}{f^l} u^{j,1} \theta_l^0 \theta_i^0 + f^l \frac{\partial_l f^i}{f^i} \frac{\partial_l f^j}{f^j} u^{j,1} \theta_l^0 \theta_i^0 \Bigg) \frac{\partial}{\partial \theta_i^s}.$$

Main result

Theorem

The cohomology $H^p_d(\hat{\mathcal{A}}[\lambda], D_\lambda)$ *vanishes for all bi-degrees* (p, d)*, unless*

$$d = 0, \dots, n, \quad p = d, \dots, d + n,$$

$$d = 2, \dots, n + 2, \quad p = d, \dots, d + n - 1.$$

Simple observation

Let (C, d) be a cochain complex with a bounded decreasing filtration

$$\cdots \subset F^{i+1}C \subset F^{i}C \subset \cdots$$

and let (E_k, d_k) be the associated spectral sequence. Then

$$H^{\ell}(E_k, d_k) = 0 \quad \Longrightarrow \quad H^{\ell}(C, d) = 0.$$

First filtration

- The degree $\deg_u$ defined by

$$\deg_u u^{i,s} = 1, \quad i = 1, \dots, n,\ s \geqslant 1$$

  and zero otherwise.
- The first filtration on $\hat{\mathcal{A}}[\lambda]$ is given by

$$F^r \hat{\mathcal{A}}^p[\lambda] = \{f \in \hat{\mathcal{A}}^p[\lambda], p + \deg_u f \geqslant r\}.$$

- Denote Δ_k the homogeneous components of D_λ on $\hat{\mathcal{A}}[\lambda]$:

$$D_\lambda = \Delta_{-1} + \Delta_0 + \dots, \qquad \deg_u \Delta_k = k.$$

- The page zero E_0 is simply given by $\hat{\mathcal{A}}[\lambda]$ with the differential Δ_{-1}.
- Explicitly:

$$\Delta_{-1} = \sum_{s \geqslant 1} (u^i - \lambda) f^i \theta_i^{1+s} \frac{\partial}{\partial u^{i,s}}.$$

Proposition

The first page is given by

$$E_1 = H(\hat{\mathcal{A}}[\lambda], \Delta_{-1}) \cong \hat{\mathcal{C}}[\lambda] \oplus \bigoplus_{i=1}^{n} \mathrm{Im}\left(\hat{d}_i : \hat{\mathcal{C}}_i \to \hat{\mathcal{C}}_i\right)$$

where

$$\hat{\mathcal{C}} := C^\infty(U)[[\theta_1^0, \dots, \theta_n^0, \theta_1^1, \dots, \theta_n^1]],$$

$$\hat{\mathcal{C}}_i := \hat{\mathcal{C}}[[\{u^{i,s}, \theta_i^{s+1}, s \geqslant 1\}]],$$

$$\hat{d}_i = \sum_{s \geqslant 1} \theta_i^{s+1} \frac{\partial}{\partial u^{i,s}} \quad \textit{(de Rham).}$$

Proof

To prove the Poincarè lemma

$$H(\hat{\mathcal{C}}_i, \hat{d}_i) = \hat{\mathcal{C}}$$

we can define an homotopy map, $i = 1, \ldots, n$, $s \geqslant 1$

$$h_{i,s} = \frac{\partial}{\partial \theta_i^{s+1}} \int du^{i,s},$$

with zero integration constant, then we have

$$h_{i,s}\hat{d}_i + \hat{d}_i h_{i,s} = 1 - \pi_{u^{i,s}} \pi_{\theta_i^{s+1}}.$$

Proof

Similarly, to prove the Proposition we use two homotopy maps. The first is

$$h_{i,s} = \sigma_i \frac{1}{u^i - \lambda} \frac{1}{f^i} \frac{\partial}{\partial \theta_i^{s+1}} \int du^{i,s}$$

which satisfies

$$\begin{aligned} h_{i,s}\Delta_{-1} + \Delta_{-1} h_{i,s} =& (1 - \pi_{u^{i,s}} \pi_{\theta_i^{s+1}})(1 - \pi_{\lambda - u^i}) + \\ & + \left(\sum_{\substack{t \geqslant 1 \\ j}} \frac{f^j}{f^i} \frac{\partial}{\partial \theta_i^{s+1}} \theta_j^{t+1} \int du^{i,s} \frac{\partial}{\partial u^{j,t}} \right) \pi_{\lambda - u^i}. \end{aligned}$$

It follows that we can kill the dependence on all the variables $u^{i,s}$, θ_i^{s+1} with $i = 1, \dots, n$, $s \geqslant 1$, in the λ-dependent part of any cocyle.

Proof

The second homotopy map is , for $i \neq j$

$$h_{i,s;j,t} = \frac{1}{u^i - u^j} \frac{1}{f^i f^j} \frac{\partial}{\partial \theta_i^{s+1}} \frac{\partial}{\partial \theta_j^{t+1}} \int du^{i,s} \int du^{j,t}$$

and we have for $\Delta_{-1} = d'' - \lambda d'$

$$[h_{i,s;j,t}, d'' d'] = (1 - \pi_{u^{i,s}} \pi_{\theta_i^{s+1}})(1 - \pi_{u^{j,t}} \pi_{\theta_j^{t+1}}) + (...)d' + (...)d'',$$

where we did not specify the last two terms since they vanish when applied on elements in $\mathrm{Ker}\, d' \cap \mathrm{Ker}\, d''$.

This allows to kill mixed terms in the λ independent part of a cocycle.

Second filtration

- The second page E_2 is given by

$$E_2 = H\left(\hat{\mathcal{B}}, \Delta_0\right),$$

$$\hat{\mathcal{B}} := \hat{\mathcal{C}}[\lambda] \oplus \bigoplus_{i=1}^{n} \mathrm{Im}\left(\hat{d}_i : \hat{\mathcal{C}}_i \to \hat{\mathcal{C}}_i\right).$$

- To compute E_2 we introduce a filtration on $\hat{\mathcal{B}}$:

$$F^r\hat{\mathcal{B}} = \{f \in \hat{\mathcal{B}}, \deg_{\theta^1} f - \deg_\theta f \leqslant -r\}.$$

- The differential splits in $\Delta_0 = \Delta_{01} + \Delta_{00} + \Delta_{0,-1}$, where Δ_{01} is the part that increases the number of θ_i^1 by one.

Explicitly:

$$\begin{aligned}
\Delta_{01} &= (-\lambda + u^i) f^i \theta_i^1 \frac{\partial}{\partial u^i} \\
&\quad + \sum_{s \geqslant 1} \frac{s+2}{2} f^i u^{i,s} \theta_i^1 \frac{\partial}{\partial u^{i,s}} \\
&\quad - \frac{1}{2} \sum_{s \geqslant 1} (-\lambda + u^j) s f^j \frac{\partial_j f^i}{f^i} u^{i,s} \theta_j^1 \frac{\partial}{\partial u^{i,s}} \\
&\quad - \frac{1}{2} (-\lambda + u^j) \partial_i f^j \theta_j^1 \theta_j^0 \frac{\partial}{\partial \theta_i^0} + \frac{1}{2} \sum_{s \geqslant 0} f^i (s-1) \theta_i^1 \theta_i^s \frac{\partial}{\partial \theta_i^s} \\
&\quad - \frac{1}{2} \sum_{s \geqslant 0} (-\lambda + u^j) f^j \frac{\partial_j f^i}{f^i} (s+1) \theta_j^1 \theta_i^s \frac{\partial}{\partial \theta_i^s} \\
&\quad + \frac{1}{2} (-\lambda + u^j) f^j \frac{\partial_j f^i}{f^i} \theta_i^1 \theta_j^0 \frac{\partial}{\partial \theta_i^0}
\end{aligned}$$

- The first page E_1' of the spectral sequence associated with the second filtration $F\hat{\mathcal{B}}$ is obtained by computing the cohomology:

$$E_1' = H(\hat{\mathcal{B}}, \Delta_{01}),$$

where

$$\hat{\mathcal{B}} := \hat{\mathcal{C}}[\lambda] \oplus \bigoplus_{i=1}^{n} \mathrm{Im}\left(\hat{d}_i : \hat{\mathcal{C}}_i \to \hat{\mathcal{C}}_i\right).$$

- The differential Δ_{01} leaves each summand invariant, hence we can compute the cohomology of each summand independently.

Vanishing of $H(\hat{\mathcal{C}}[\lambda], \Delta_{01})$

- The possible monomials in $\hat{\mathcal{C}}$ are

$$\theta^0_{i_1} \cdots \theta^0_{i_k} \theta^1_{j_1} \cdots \theta^1_{j_l}.$$

- Hence the cohomology $H^p_d(\hat{\mathcal{C}}[\lambda], \Delta_{01})$ vanishes, unless

$$d = 0, \ldots, n, \quad p = d, \ldots, d + n.$$

Third filtration

- Finally we need to compute, for fixed $i = 1, \ldots, n$:

$$H\left(\hat{\mathcal{B}}_i, \Delta_{01}\right),$$

where

$$\hat{\mathcal{B}}_i := \operatorname{Im}\left(\hat{d}_i : \hat{\mathcal{C}}_i \to \hat{\mathcal{C}}_i\right).$$

- We introduce a filtration on $\hat{\mathcal{B}}_i$ by:

$$F^r\hat{\mathcal{B}}_i = \{f \in \hat{\mathcal{B}}_i, \deg_{\theta_i^1} f - \deg_\theta f \leqslant -r\}$$

- Denote by $\theta_i^1 \mathcal{D}_i$ the part of Δ_{01} that increases the degree in θ_i^1.

$$\mathcal{D}_i := \sum_{s \geqslant 1} \frac{s+2}{2} f^i u^{i,s} \frac{\partial}{\partial u^{i,s}} + \sum_{s \geqslant 2} \frac{s-1}{2} f^i \theta_i^s \frac{\partial}{\partial \theta_i^s}$$
$$- \frac{1}{2} f^i \theta_i^0 \frac{\partial}{\partial \theta_i^0} + \frac{1}{2} \sum_{j=1}^{n} (u^j - u^i) f^j \frac{\partial_j f^i}{f^i} \theta_j^0 \frac{\partial}{\partial \theta_i^0}$$

- The first page E_1'' of the spectral associated with the third filtration is obtained by computing the cohomology:

$$H(\hat{\mathcal{B}}_i, \theta_i^1 \mathcal{D}_i).$$

- Finally we can obtain the vanishing of the cohomology that implies the main theorem:

Proposition

The cohomology $H_d^p(\hat{\mathcal{B}}_i, \theta_i^1 \mathcal{D}_i)$ vanishes for all bi-degrees (p, d) unless

$$d = 2, \ldots, n+2, \quad q = d, \ldots, d+n-1.$$

Conclusions and open problems

- We compute the full bihamiltonian cohomology of the dispersionless KdV Poisson pencil.
  1. Prove conjecture of Liu-Zhang.
  2. Rederive known results on cohomology of low degree.
- We compute the full bihamiltonian cohomology of the general scalar Poisson pencil of hydrodynamic type.
- We prove a vanishing theorem for the bihamiltonian cohomology of a semisimple n component Poisson pencil of hydrodynamic type.

  $\Longrightarrow$ the dispersive deformations of these Poisson pencils are unobstructed.
- Open problem: compute the remaining entries of $BH^p_d(\Lambda_{loc}, d_1, d_2)$.
- Work in progress: Several independent variables (G.C., M. Casati, S. Shadrin.)