Orthogonal instantons and skew-Hamiltonian matrices

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Moduli Spaces
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The moduli space $M(r, n)$

$M(r, n)$= the moduli space of (slope) stable v.b. on $\mathbb{P}^2$ with Chern classes $(0, n)$ and rank $2 \leq r \leq n$. ($M(r, n) = \emptyset$ if $r > n$.)

$E \in M(r, n)$ is the cohomology bundle of the monad:

$$I \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{g} V^* \otimes \Omega^1_{\mathbb{P}^2}(2) \xrightarrow{f} V \otimes \mathcal{O}_{\mathbb{P}^2}(1)$$

$V = \mathrm{H}^1(E(-1))$ has dim n (independently from r!), $I = \mathrm{H}^1(E(-3))$ has dim $n - r$, $f \in U \otimes V \otimes V$ is the natural multiplication map, $\mathbb{P}^2 = \mathbb{P}(U)$.

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Rem: equivalently E is the cohomology of a *linear monad*:

$$V^* \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\alpha} K \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\beta} V \otimes \mathcal{O}_{\mathbb{P}^2}(1)$$

where $K = \mathrm{H}^1(E \otimes \Omega^1_{\mathbb{P}^2})$ has dim $2n + r$.

So all elements of $M(r, n)$ are "generalized instantons".

Orthogonal and symplectic bundles

- An orthogonal v.b. is a pair (E, α) consisting of a v.b. E and an iso $\alpha : E \to E^*$ with ${}^t\alpha = \alpha$.
- A symplectic v.b. is a pair (E, α) as above, with ${}^t\alpha = -\alpha$.
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Lemma. The map f encodes all the info: $E(f)$ and $E(f')$ simple are iso iff f and f' are $\mathrm{SL}(V)$-equivalent.

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We can recover the structure of the v.b. from $f \in U \otimes V \otimes V$.

Proposition. The bundle $E(f)$ is:

- orthogonal iff the map $f \in U \otimes \Lambda^2 V$;
- symplectic iff the map $f \in U \otimes S^2 V$.

Unstructured and symplectic bundles

$M(r,n)$= the moduli space of stable unstructured v.b. on $\mathbb{P}^2$ with Chern classes $(0,n)$ and rank $2 \leq r \leq n$.

Theorem [Hulek, 1980] When non-empty, $M(r,n)$ is a smooth irreducible variety of dimension $2rn - r^2 + 1$.

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$M_{sp}(r,n)$ = the moduli space of stable symplectic v.b. on $\mathbb{P}^2$ with Chern classes $(0,n)$ and rank $2 \leq r \leq n$.

Theorem [Ottaviani, 2007] When non-empty, $M_{sp}(r,n)$ is a smooth irreducible variety of dimension $(r+2)n - \binom{r+1}{2}$.

Rem: in particular r is even in the symplectic case.

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On the contrary, the irreducibility argument **does not**.
Why? What goes wrong? What goes right?
(Results from a joint project with **R. Abuaf**.)

Bundles with trivial splitting on a line

Notation: $M_\star(r,n)$ for $\star = \emptyset$, sp, ort.

$$M^0_\star(r,n) = \{E \in M_\star(r,n) \mid E_{|_\ell} = \mathcal{O}^r_{\mathbb{P}^1} \text{ for some line } \ell\}$$

By semicontinuity, if $E_{|_\ell}$ is trivial on a line ℓ, then it is trivial on the general line.

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Why? Proving irreducibility of $M^0_\star(r,n)$ is easier!

Moreover, using a deformation argument (Hirschowitz) we get:

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- $M_{ort}(r,n) = \ldots$ **nope!**

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- $M_{sp}(r,n) = \overline{M^0_{sp}(r,n)}$ is irreducible [Ottaviani, 2007]
- $M_{ort}(r,n) \neq \overline{M^0_{ort}(r,n)}$

Degeneration arguments and the Mumford invariant

When we restrict an unstructured or a symplectic bundle to $\mathbb{P}^1$, the only rigid bundle is the trivial one, $\mathcal{O}_{\mathbb{P}^1}^r$ [Ramanathan, 1983].

In the orthogonal case instead there are 2 rigid bundles:

$$\mathcal{O}_{\mathbb{P}^1}^r \quad \text{and} \quad \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^{r-2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

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- If E is an orthogonal v.b. on $\mathbb{P}^1$, then $\mathrm{h}^1(E(-1)) \bmod 2$ is invariant under deformations. [Mumford, 1971]
- Orthogonal rk 2 v.b. on $\mathbb{P}^1$ are rigid; for rk ≥ 3 the *Mumford invariant* is the only one. Two such v.b. can be deformed into each other iff they have the same Mumford inv. [Hulek, 1981]

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Remember that for us $\mathrm{h}^1(E(-1)) = n = \dim V = c_2(E)$.

Proposition. If $E \in M^0_{ort}(r, n)$, then n is even.

A closer look

If P, Q and R the 3 $n \times n$ “slices” of $f \in U \otimes V \otimes V$ then:

$$\mathrm{H}^0(f) = \begin{bmatrix} 0 & P & Q \\ -P & 0 & R \\ -Q & -R & 0 \end{bmatrix}$$

P, Q and R are $n \times n$ unstructured, symmetric, and skew-symmetric matrices for $\star = \emptyset$, *sp*, and *ort* respectively.

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Lemma. $E \in M^0_\star(r, n)$ iff Q is invertible.

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Since we can prove independently that $\mathrm{rk}(\mathrm{H}^0(f)) = 2n + r$, we get a link between $r = \mathrm{rk}\, E$ and the 3 matrices P, Q, and R:

$$\mathrm{rk}(PQ^{-1}R - RQ^{-1}P) = r$$

Irreducibility results

Using a standard fibration argument (so standard that we skip it):

Theorem. Let n and $3 \leq r \leq n$ be two positive integers, n even. Let V be a complex v.s. of dimension n, and let J denote the standard symplectic form. If the variety:

$$\mathcal{C}_{r,n} = \{(A, B) \in \Lambda^2 V \times \Lambda^2 V \mid \mathrm{rk}(AJB - BJA) \leq r\}$$

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Key Lemma. The variety $\mathcal{C}_{r,n}$ is irreducible for some (TBD) values of r and n.

Skew-Hamiltonian matrices

What happens for $M^0(r,n)$ and $M^0_{sp}(r,n)$? With similar arguments one reduces to proving the irreducibility of:

$$\{(A,B) \in (V \otimes V) \times (V \otimes V) \mid \mathrm{rk}[A,B] \leq r\}$$

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Notice that $\mathrm{rk}(AJB - BJA) = \mathrm{rk}[JA, JB]$.
Enter the scene: skew-Hamiltonian matrices.

A *skew-Hamiltonian* matrix is of the form JB, $B \in \Lambda^2 V$.

Rem: skew-Hamiltonians are the less cool cousins of Hamiltonians (JS, $S \in S^2V$), which correspond to the Lie algebra of $\mathrm{Sp}(n)$.

Regular elements

Remember: we are studying pairs (A, B) s.t. $\mathrm{rk}(AJB - BJA) \leq r$.
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So we want regular matrices too! Alas, the notion of regular element is meaningless for us...

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Moreover the two actions are consistent with each other:

$$M * (JB) = J(M \star B)$$

This allows us to define *regular skew-Hamiltonians* and prove that they have the smallest possible J-commutator.

Diamond matrices

Not only can we estimate $\dim(\operatorname{Im} \varphi^B)$ for a regular B, we can also get **explicit equations**!

Proposition.[Noferini, 2013] Let $B \in \Lambda^2 V$ s.t. JB is a regular skew-Hamiltonian, and let $A \in \Lambda^2 V$ any skew-symmetric matrix. Then $AJB - BJA$ is symplectically congruent to a *diamond matrix*.

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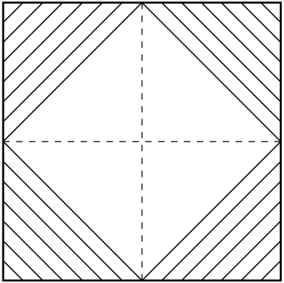
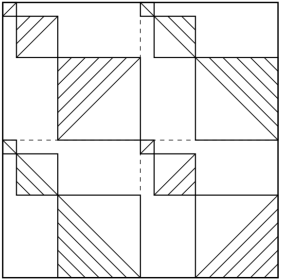


A diamond matrix corresponding to the partition $\underline{d} = (\frac{n}{2})$.



A diamond matrix corresponding to a partition $\underline{d} = (d_1, d_2, d_3)$.

Irreducibility results and conjectural bounds

With some elbow grease we finally prove irreducibility, by means of a strong connectedness result.

Lemma. If JB is regular, then the intersection $\operatorname{Im}\varphi^B \cap S^2 V_{\leq r}$ is irreducible of dimension $nr - \frac{3}{2}n - \binom{r}{2}$ for $r = n$ and $n \geq 4$ and for $r = n - 1$ and $n \geq 8$.

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In fact **we believe that more is true**:

Conjecture. Irreducibility of $M^0_{ort}(r, n)$ holds for any $6r - 5n \geq 2$.

Rem: For $r = n$ and $n - 1$ we re-obtain what we just proved.

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- In fact for n odd $M_{ort}(r,n) = \overline{M^1_{ort}(r,n)}$, where:

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  We call these bundles *weakly framed*.
- The moduli space is not empty! $(S^2\,\mathrm{T}\,\mathbb{P}^2)(-3)$, is an example of a weakly framed stable rk 3 orthogonal bundle on $\mathbb{P}^2$ with Chern classes $(0,3)$.

A curious result and one last question

Proposition. Let V be a complex vector space of even dimension n. Define ϕ as:

$$\phi : \mathbb{P}(\Lambda^2 V \times \Lambda^2 V) \dashrightarrow \mathbb{P}(S^2 V) \qquad ([A],[B]) \mapsto [AJB - BJA].$$

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b) For $n = 4$, $\operatorname{Im}\phi$ is a $\mathbb{G}(2,5)$ in $\mathbb{P}^9 = \mathbb{P}(S^2\mathbb{C}^4)$.
c) For $n = 6$, $\operatorname{Im}\phi$ is a hypersurface of deg 4 in $\mathbb{P}^{20} = \mathbb{P}(S^2\mathbb{C}^6)$.

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$$\phi : \mathbb{P}(\Lambda^2 V \times \Lambda^2 V) \dashrightarrow \mathbb{P}(S^2 V) \qquad ([A],[B]) \mapsto [AJB - BJA].$$

a) For $n = 2$, ϕ is not defined.

b) For $n = 4$, $\operatorname{Im}\phi$ is a $\mathbb{G}(2,5)$ in $\mathbb{P}^9 = \mathbb{P}(S^2\mathbb{C}^4)$.

c) For $n = 6$, $\operatorname{Im}\phi$ is a hypersurface of deg 4 in $\mathbb{P}^{20} = \mathbb{P}(S^2\mathbb{C}^6)$.

d) For $n \geq 8$, ϕ is locally of maximal rank. (So ϕ is dominant.)

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Question. Is there a value of r for which ϕ composed with the projection $\mathbb{P}(S^2 V) \twoheadrightarrow \mathbb{P}(S^2 V_{\leq r})$ is surjective? This is false for $r = 2$ [Noferini, 2013], but it remains open for higher values of r.

Thank you :)