Hitchin Pairs On A Singular Curve

Pabitra Barik

Workshop on Moduli spaces in algebraic geometry and physics,Chern Institute Of mathematics

September 15, 2015

Hitchin pairs on a smooth curve

We first recall some work on Hitchin pairs on a smooth curve.

Hitchin pairs on a smooth curve

We first recall some work on Hitchin pairs on a smooth curve.

Let L be a fixed line bundle on a compact Riemann surface X of genus $g \geq 2$.

Hitchin pairs on a smooth curve

We first recall some work on Hitchin pairs on a smooth curve.

Let L be a fixed line bundle on a compact Riemann surface X of genus $g \geq 2$.

Definition

A semistable (stable) Hitchin pair is a vector bundle E on X together with a morphism $\phi : E \to E \otimes L$ such that $\mu(F) \leq \mu(E), (resp. \lneq)$ $\forall$ nontrivial ϕ -invariant subbundle $F \subset E$, where $\mu() = \frac{deg()}{rank()}$.

Theorem

(N.J. Hitchin[1987], Nitin Nitsure[1991])

1. The coarse moduli space $\mathcal{M}(r, d, L)$ of S-equivalence classes of semistable Hitchin pairs of rank r, degree d, exists as a quasi-projective scheme. The isomorphism classes of stable Hitchin pairs is an open subscheme $\mathcal{M}^s(r, d, L)$ of $\mathcal{M}(r, d, L)$.

Theorem

(N.J. Hitchin[1987], Nitin Nitsure[1991])

1. The coarse moduli space $\mathcal{M}(r,d,L)$ of S-equivalence classes of semistable Hitchin pairs of rank r, degree d, exists as a quasi-projective scheme. The isomorphism classes of stable Hitchin pairs is an open subscheme $\mathcal{M}^s(r,d,L)$ of $\mathcal{M}(r,d,L)$.
2. The Hitchin map from $\mathcal{M}(r,d,L)$ into $\mathcal{A} := \bigoplus_{i=1}^{r} H^0(L^i)$ defined by characteristic polynomial, is proper.

Theorem

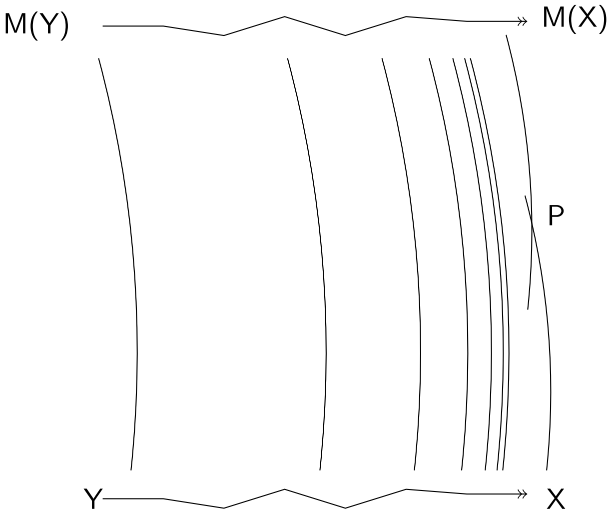
(N.J. Hitchin[1987])

The generic Hitchin fiber, $h^{-1}(s) \cong Pic^{\delta}(C_s)$, where C_s is the spectral curve determined by the generic point $s \in \mathcal{A}$

Nagaraj-Seshadri's Work

Let Y be a smooth irreducible projective curve and $M(Y)$ be the moduli space of semistable vector bundles (of fixed rank and degree) on Y.

One Approach to study M(Y):



Nagaraj-Seshadri consider the case of a reducible projective curve $X = X_1 \cup X_2$ having only two smooth irreducible components X_1 and X_2 meeting transversally at one point p.

Nagaraj-Seshadri consider the case of a reducible projective curve $X = X_1 \cup X_2$ having only two smooth irreducible components X_1 and X_2 meeting transversally at one point p.

- By a torsion free sheaf we mean a pure sheaf of dimension 1.

The main result Nagaraj-Seshadri says that

- The moduli space $M(X)$ of semistable torsion free sheaves on X of rank 2 and fixed Euler characteristic χ has only two irreducible components, M_1 and M_2. (C.S. Seshadri)

The main result Nagaraj-Seshadri says that

- The moduli space $M(X)$ of semistable torsion free sheaves on X of rank 2 and fixed Euler characteristic χ has only two irreducible components, M_1 and M_2. (C.S. Seshadri)
- Further, if χ is odd, M_1 and M_2 are smooth varieties intersecting in a smooth variety N.

The main result Nagaraj-Seshadri says that

- The moduli space $M(X)$ of semistable torsion free sheaves on X of rank 2 and fixed Euler characteristic χ has only two irreducible components, M_1 and M_2. (C.S. Seshadri)
- Further, if χ is odd, M_1 and M_2 are smooth varieties intersecting in a smooth variety N.
- M_1 and M_2 are isomorphic to suitable moduli spaces of vector bundles on the irreducible components of X, while N turns out to be a product of certain moduli spaces of parabolic bundles on X_1 and X_2 respectively.

Goal of This Talk

- Let X be a reducible projective curve, $X = X_1 \cup X_2$ with a single node at a point $p \in X_1 \cap X_2$, where X_1 and X_2 are two smooth curves of genus g_1 and g_2 respectively.

Goal of This Talk

- Let X be a reducible projective curve, $X = X_1 \cup X_2$ with a single node at a point $p \in X_1 \cap X_2$, where X_1 and X_2 are two smooth curves of genus g_1 and g_2 respectively.
  The aim of this talk is to construct moduli space of torsion free Hitchin pairs over X, describe the moduli space, define Hitchin map and describe the generic Hitchin fiber.

Moduli Problem of torsion free Hitchin pairs

Let L be a fixed line bundle over X. One knows that L is obtained by giving line bundles L_i on $X_i, \ i = 1, 2$ together with a gluing isomorphism $l : L_{1,p} \simeq L_{2,p}$.

A polarization on X is a choice of rational weights $0 \lneq a_1, a_2 \lneq 1$ with $a_1 + a_2 = 1$. We fix a polarization $a = (a_1, a_2)$ and we adopt the following definition of semistability:

A polarization on X is a choice of rational weights $0 \lneq a_1, a_2 \lneq 1$ with $a_1 + a_2 = 1$. We fix a polarization $a = (a_1, a_2)$ and we adopt the following definition of semistability:

Definition

For a torsion free sheaf $\mathcal{F}$ of type (r_1, r_2) on X, we define rank $r := a_1 r_1 + a_2 r_2$ and it's slope $\mu(\mathcal{F}) := \frac{\chi(\mathcal{F})}{r}$, where $\chi(\mathcal{F}) = dim H^0(\mathcal{F}) - dim H^1(\mathcal{F})$.

A semistable (resp. stable) torsion free Hitchin pair $(\mathcal{F}, \phi)$ consists of a torsion free sheaf $\mathcal{F}$ on X and a morphism $\phi : \mathcal{F} \to \mathcal{F} \otimes L$ such that $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$ (resp.$\lneq$) for all nontrivial proper subsheaf $\mathcal{G} \subset \mathcal{F}$ with $\phi(\mathcal{G}) \subset \mathcal{G} \otimes L$.

We consider only rank 2 torsion free Hitchin pairs for the moduli problem .

Definition

A family of a torsion free Hitchin pairs parametrized by a scheme T is a pair (E_T, ϕ_T) consists of a coherent sheaf E_T over $X \times T$ which is flat over T and a morphism $\phi_T : E_T \to E_T \otimes \pi_X^* L$ such that (E_t, ϕ_t) is a torsion free Hitchin pair $\forall\, t \in T$.

Definition

Two families $(E_T, \phi_T), (E^1_T, \phi^1_T)$ are said to be equivalent if and only if $(E_t, \phi_t) \cong (E^1_t, \phi^1_t)$, $\forall t \in T$ i.e there exists a sheaf isomorphism $h_t : E_t \to E^1_t$ such that the following diagram commutes:

$$\begin{array}{ccc} E_t & \xrightarrow{\phi_t} & E_t \otimes L \\ \Big\downarrow h_t & \circlearrowleft & \Big\downarrow h_t \otimes Id \\ E^1_t & \xrightarrow{\phi^1_t} & E^1_t \otimes L \end{array}$$

$\forall t \in T$

We have the following theorem

Theorem

The coarse moduli space $\mathcal{M}(2, \chi, L)$ of S-equivalence classes of semistable torsion free Hitchin pairs (E, ϕ) on X of rank 2, Euler characteristic χ, exists as a quasi-projective scheme. The isomorphism classes of stable torsion free Hitchin pairs is an open subscheme $\mathcal{M}^s(2, \chi, L)$ of $\mathcal{M}(2, \chi, L)$.

Hitchin Triples

We fix a direction for "Hitchin triples" i.e we only consider triples of the form $(\hat{V}, \hat{W}, \overrightarrow{A})$. Analogous definitions and results come for triples in the opposite direction.

Definition

A Hitchin triple *is a triple* $(\hat{V}, \hat{W}, \overrightarrow{A})$ *consists of two Hitchin pairs* $\hat{V} := (V, \phi)$ *on* X_1 *and* $\hat{W} := (W, \psi)$ *on* X_2 *respectively together with a linear map* $\overrightarrow{A} : V_p \to W_p$ *such that the following diagram commutes:*

$$\begin{array}{ccc} V_p & \xrightarrow{\phi_p} & V_p \otimes L_{1,p} \\ \downarrow{\scriptstyle \overrightarrow{A}} & \circlearrowleft & \downarrow{\scriptstyle \overrightarrow{A} \otimes l} \\ W_p & \xrightarrow{\psi_p} & W_p \otimes L_{2,p} \end{array}$$

Fix a polarization (a_1, a_2) on the nodal curve X with $a_i \geq 0$, are positive rational numbers with $a_1 + a_2 = 1$.

Fix a polarization (a_1, a_2) on the nodal curve X with $a_i \geq 0$, are positive rational numbers with $a_1 + a_2 = 1$.

We define the (a_1, a_2)-slope of a Hitchin triple $(\hat{V}, \hat{W}, \overrightarrow{A})$ as follows:

$$\mu(\hat{V}, \hat{W}, \overrightarrow{A}) = \mu_{(a_1,a_2)}(\hat{V}, \hat{W}, \overrightarrow{A}) := \frac{\chi(\hat{V}, \hat{W}, \overrightarrow{A})}{r_1 a_1 + r_2 a_2}$$

where r_1, r_2 are ranks of the vector bundles V, W respectively

and

$$\chi(V) = deg(V) + r_1(1 - g_1), \chi(W) = deg(W) + r_2(1 - g_2)$$

$$\chi(\hat{V}, \hat{W}, \overrightarrow{A}) := \chi(V) + \chi(W) - r_2$$

Definition

A Hitchin triple $(\hat{V}^1, \hat{W}^1, \overrightarrow{A^1})$ *is said to be a* sub Hitchin triple of $(\hat{V}, \hat{W}, \overrightarrow{A})$ if $\hat{V}^1$ is sub Hitchin pair of $\hat{V}$, and $\hat{W}^1$ is sub Hitchin pair of $\hat{W}$ and the following diagram commutes:

$$\begin{array}{ccc} V_p^1 & \xrightarrow{\;inclusion\;} & V_p \\ \Big\downarrow{\scriptstyle \overrightarrow{A^1}} & \circlearrowleft & \Big\downarrow{\scriptstyle \overrightarrow{A}} \\ W_p^1 & \xrightarrow{\;inclusion\;} & W_p \end{array}$$

Definition

A Hitchin triple $(\hat{V}, \hat{W}, \overrightarrow{A})$ *is said to be* (a_1, a_2)-semistable (resp. stable) if $\mu(\hat{V}^1, \hat{W}^1, \overrightarrow{A^1}) \leq \mu(\hat{V}, \hat{W}, \overrightarrow{A})$ for all nontrivial proper sub-Hitchin triples $(\hat{V}^1, \hat{W}^1, \overrightarrow{A^1})$ (resp. $\lneq$).

Definition

A Hitchin triple $(\hat{V}, \hat{W}, \overrightarrow{A})$ *is said to be* (a_1, a_2)-semistable (resp. stable) if $\mu(\hat{V}^1, \hat{W}^1, \overrightarrow{A^1}) \leq \mu(\hat{V}, \hat{W}, \overrightarrow{A})$ for all nontrivial proper sub-Hitchin triples $(\hat{V}^1, \hat{W}^1, \overrightarrow{A^1})$ (resp. $\lneq$).

Theorem

$(\mathcal{E}, \theta) \mapsto ((V, \phi), (W, \psi), \overrightarrow{A})$ *is an equivalence of categories between category of torsion free Hitchin pairs on* X *and Hitchin triples over* X*. This equivalence identifies* a*-semistable (resp.* a*-stable)torsion free Hitchin pairs with* a*-semistable (resp.* a*-stable) Hitchin triples.*

- For the rest of this talk, we assume $a_1\chi \notin \mathbb{Z}$ and Euler characteristic χ is odd. This technical non-degeneracy condition ensures that with χ odd we have semistability $=$ stability.

Let

$$\mathcal{M}_1 = \mathcal{M}(2, a_1, \overrightarrow{\chi})$$

be the moduli space (a_1, a_2)-semistable Hitchin triples $(\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2, \overrightarrow{A})$ of rank 2 with

$a_1\chi \lneq \chi_{X_1}(\mathcal{F}_1) \lneq a_1\chi + 1$

$a_2\chi + 1 \lneq \chi_{X_2}(\mathcal{F}_2) \lneq a_2\chi + 2$ and

$\chi_X(\mathcal{F}) + 2 = \chi_{X_1}(\mathcal{F}_1) + \chi_{X_2}(\mathcal{F}_2).$

Similarly, let

$$\mathcal{M}_2 = \mathcal{M}(2, a_1, \overleftarrow{\chi})$$

be the moduli space (a_1, a_2)-semistable Hitchin triples $(\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2, \overleftarrow{B})$ of rank 2 with

$a_1\chi + 1 \lneq \chi_{X_1}(\mathcal{F}_1) \lneq a_1\chi + 2$

$a_2\chi \lneq \chi_{X_2}(\mathcal{F}_2) \lneq a_2\chi + 1$ and

$\chi_X(\mathcal{F}) + 2 = \chi_{X_1}(\mathcal{F}_1) + \chi_{X_2}(\mathcal{F}_2)$.

$$\mathcal{N}_1 = \{[(\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2, \overrightarrow{A})] \in \mathcal{M}_1^H \mid rk(\overrightarrow{A}) = 1\}$$

and

$$\mathcal{N}_2 = \{[(\hat{\mathcal{F}}_1^1, \hat{\mathcal{F}}_2^1, \overleftarrow{B})] \in \mathcal{M}_2^H \mid rk(\overleftarrow{B}) = 1\}$$

are closed subschemes of $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively.

There is a natural isomorphism $\sigma : \mathcal{N}_1 \to \mathcal{N}_2$.

Theorem

With the above notations and definitions,if $a_1\chi \notin \mathbb{Z}$,and χ be an odd integer then, $\mathcal{M}(2,(a_1,a_2),\chi) = \mathcal{M}_1 \cup \mathcal{M}_2$, with the above identification σ of closed subschemes $\mathcal{N}_1$ of $\mathcal{M}_1$ with $\mathcal{N}_2$ of $\mathcal{M}_2$.

Definition

(The Hitchin map) *Define the characteristic polynomial of a Hitchin triple* $(\hat{V}, \hat{W}, \overrightarrow{A})$ *as a pair* (f_V, f_W)*, where* f_V *is the characteristic polynomial of the Higgs structure* $\phi : V \to V \otimes L_1$ *on the curve* X_1 *and* f_W *is the characteristic polynomial of the Higgs structure* $\psi : W \to W \otimes L_2$ *on the curve* X_2*. Define the Hitchin map* $\mathsf{h} : \mathcal{M}(2, \chi, \mathsf{L}) \to \mathsf{A}(\mathsf{L})$ *from the moduli space of Hitchin triples* $\mathcal{M}(2, \chi, L)$ *to* $A(L) = \mathcal{A}_1 \times \mathcal{A}_2$*, where* $\mathcal{A}_i = H^0(L_i) \oplus H^0(L_i^2), \ i = 1, 2$ *as*

$$\mathsf{h}(\hat{\mathsf{V}}, \hat{\mathsf{W}}, \overrightarrow{A}) := (\mathsf{f}_\mathsf{V}, \mathsf{f}_\mathsf{W})$$

Properness Of Hitchin Map

Theorem

The Hitchin map $\mathsf{h} : \mathcal{M}(2, \chi, \mathsf{L}) \to \mathsf{A}(\mathsf{L})$ *is proper.*

Spectral Curve

$$\mathcal{A}(p) := \{(s_1, s_2) \in \mathcal{A}_1 \times \mathcal{A}_2 \mid s_1(p) = s_2(p)\}$$

Then $\mathsf{h}(\mathcal{M}) \subset \mathcal{A}(p)$.

Spectral Curve

$$\mathcal{A}(p) := \{(s_1, s_2) \in \mathcal{A}_1 \times \mathcal{A}_2 \mid s_1(p) = s_2(p)\}$$

Then $\mathrm{h}(\mathcal{M}) \subset \mathcal{A}(p)$.

Further, we know $\mathrm{h} : \mathcal{M} \to \mathcal{A}(p)$ is proper.

Let,

$$\mathcal{A}^{ur} := \{s \in \mathcal{A}(p) \mid Y_i \ smooth, \ Y_i \to X_i \ unramified \ at \ p\}.$$

$\mathcal{A}^{ur}$ is an *open* subset of $\mathcal{A}(p)$.

Let,

$$\mathcal{A}^{ur} := \{s \in \mathcal{A}(p) \mid Y_i \ smooth, \ Y_i \to X_i \ unramified \ at \ p\}.$$

$\mathcal{A}^{ur}$ is an *open* subset of $\mathcal{A}(p)$.

PROPOSITION

Let L be a line bundle on X such that the linear system $|L^2|$ is base-point free. Let $s \in \mathcal{A}^{ur}$. Then, s defines a spectral curve $C = Y_1 \cup Y_2$, where $\#(C_{sing}) = 2$, i.e the union of two smooth curves meeting transversally at 2 points.

Analogue of classical Hitchin theorem

Theorem

Let $s \in \mathcal{A}^{ur}$ and let $\pi : C_s \to X$ be the associated spectral cover of X.

We have an isomorphism $\overline{P^{\gamma}(C_s)} \simeq \mathsf{h}^{-1}(s)$ between the Oda-Seshadri compactification $\overline{P^{\gamma}(C_s)}$ of the Picard variety of line bundles on C_s of degree γ and the generic Hitchin fiber.

THANK YOU!